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ABSTRACT

This commentary is designed for use with "Unified Modern Mathematics, Course II," Parts 1 and 2. As in the commentary for "Course I," statements of the specific purposes and goals of each section of every chapter are presented. Also included are suggestions for teaching the concepts presented in each section, time estimates for each section, suggested instructional aids to presenting various concepts, and references for further study. Chapter examinations are provided which constitute comprehensive tests for each chapter. [Not available in hardcopy due to marginal legibility of original document.] (FL)

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Secondary School Mathematics
Curriculum Improvement Study

UNIFIED MODERN MATHEMATICS

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COURSE II

TEACHERS COMMENTARY

TEACHERS COLLEGE COLUMBIA UNIVERSITY

SE 010 734

ED0 46773

TEACHERS COMMENTARY
FOR
COURSE II
OF
UNIFIED MODERN MATHEMATICS

TEACHERS COLLEGE
COLUMBIA UNIVERSITY

SECONDARY SCHOOL MATHEMATICS
CURRICULUM IMPROVEMENT STUDY

Table of Contents

Page
How To Use This Commentary....1
Time Estimation - Course II....6
Errata For Course II....7
Chapter 1....14
Chapter 2....43
Chapter 3....74
Chapter 4....132
Chapter 5....154
Chapter 6....174
Chapter 7....217
Chapter 8....254
Chapter 9....285
Chapter 10....340
Appendix A....361

Secondary School Mathematics Curriculum Improvement Study

Course II - Teacher Commentary

HOW TO USE THIS COMMENTARY

1. Purposes. At the start of the commentary for each chapter, the overall purposes and goals for the chapter are stated. Often, specific sections within the chapter are identified here as they would relate to each purpose stated. Similarly, the commentary for every section within the chapter will begin with a statement of specific purposes.

2. Sections. There are two basic types of sections within each chapter. One type presents concepts; the second type consists of exercises. The sections have been ordered so that a section (or sometimes two sections) of exposition is followed by a section of related exercises. Within various sections, the teacher will find: possible motivational devices; a variety of approaches; notations relative to difficult exercises; suggestions for placement of exercises as class work; homework or self-study; hints regarding difficulties that may occur; new vocabulary underscored; and some abstract background for the teacher.

3. Time Estimates. In terms of days, a time estimate will be found at the beginning of each chapter commentary. This is the estimate for the chapter; it is based upon

individual time estimates for sections within the chapter.

Time estimates are given only to those sections containing some form of exposition. It is assumed that each exercise section is to be grouped with the concept section immediately preceding it relative to time estimations.

The teacher should note that Chapter 11 is not to be included as part of Course II. Also, the teacher should feel free to assign Appendix A as a self-study unit, and pace his teaching so that emphasis is placed on the chapters 1 to 10.

4. Exercises. Certain exercises have proved to be more successful when discussed within the actual lesson rather than assigned as homework. Suggestions regarding the placement of exercises appear at various points within the commentary.

The teacher need not hold rigidly to the exercises as listed. He is free to choose, add or alter any exercises whatsoever. In instances stressing drill, the teacher may wish to select or limit exercises depending upon the particular skills of his class and/or individual students. Difficult problems have been starred and may be considered as optional. However, these problems are the most rewarding as well as the most challenging, and the teacher should discuss some of these in the classroom and/or assign them to the better students as homework. In all instances, the teacher should study the exercises **before assigning** them, carefully noting the concepts involved and *approximating* the *time* required for those exercises chosen. To insure that the teachers' evaluation of time for an assignment is as accurate as possible, the teachers should occasionally ask students to time homework assignments, allowing him to compare the true mean time with his judgement.

5. **Proofs.** The proofs presented in the commentary and the text are not to be accepted as the only possible, logical proof. The teacher should expose the students to other approaches, and *encourage* the students to develop their own proofs. Student approaches, very often, are more direct, less involved, yet complete mathematical solutions to problems.

6. **Self-Study Units.** At various points within a chapter, certain sections will be identified as "self-study" ones. (These are fewer in number in comparison to Course I.) In essence, these sections usually contain simple applications of concepts previously taught and such sections should be regarded as being within the scope of each student's ability.

7. **Summary and Review Exercises.** At the end of each chapter, the teacher will find a summary of the main concepts studied, followed by a series of related review exercises. The teacher may wish to assign the reading of the summary and the completion of the review exercise as:

(a) homework to be reviewed in class the following day,

(b) self-study with time allowed the following day for student questions,

(c) classwork or

(d) test items.

8. **Tests**. At the end of each chapter commentary, the teacher will find a series of suggested test items. The teacher should again feel free to choose, add, or alter any of these problems in constructing a test for his own class. An additional source of test items, when altered, would be the review exercises appearing at the end of each chapter in the text.

9. **Unified Approach**. The teacher should be alert to related topics and concepts throughout the entire course. The students should be able to grasp key ideas that weave a continual thread throughout the main body of the text of Course II. (Many of these concepts were previously developed in Course I.) Properties and relations must continually be placed in the foreground and mathematics should be viewed as a united subject rather than a series of disjoint branches of learning.

Teachers Commentary of Unified Modern Mathematics Course II is an expansion of the original commentary written by the authors of the text. It was revised by the following pilot teachers in the SSMCIS Project:

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It is hoped that the teaching experience of this team will be reflected in a practical list of suggestions and a reasonable estimation of time allotments for the whole of this commentary.

Time Estimation - Course II

Chapters	Teaching Days	Test	Total
Chapter 1	13 days	1	14
Chapter 2	16 - 17 days	1	17 - 18
Chapter 3	18 - 21 days	1	19 - 22
Chapter 4	15 days	1	16
Chapter 5	14 days	1	15
Chapter 6	19 - 23 days	1	20 - 24
Chapter 7	14 days	1	15
Chapter 8	14 days	1	15
Chapter 9	25 - 28 days	1	26 - 29
Chapter 10	11 days	1	12

Appendix A - Independent Study - no time estimate

ERRATA FOR COURSE II

PART I

Chapter 2

Page 109	Line 21	$\underline{a = 3}$ instead of $a = 0$

Chapter 3

Page	Line	Change
216	last line of first paragraph	should read: "that $x = \frac{5}{7}$ implies that $7x + 10 = 15$."
223	section 8(a)	should read: "$3x^2 - 14x + 8 = 0$"

Chapter 5

Page	Line	Change
227	5	expension ⟶ extension
	8	and $x \cdot a = b$ may have no whole number solutions.
	16	$x^2 + 6x + 8 \underline{\ =\ } 0$ has as the
228	3	than ⟶ that
	20	$5 \longrightarrow 5^2$
242	line 5 from bottom	- - - ⟶ ...
245	(an inclusion)	

At this point in the discussion of real numbers, the teacher should introduce the following procedure in order that the student will be able to find a rational number of the form $\frac{a}{b}$ $a, b \in Z$ $b \neq 0$ when he is given a repeating decimal form.

Example 1. Let ① $N = .3\overline{3}$
Mult. ① X 10 ②. $10N = 3.3\overline{3}$
Subt. ① from ② $9N = 3.0\overline{0}$
$N = \frac{3}{9}$
$N = \frac{1}{3}$

Example 2. Let ① $N = .121\overline{2}$
Mult. ① X 100 ②. $100N = 12.\overline{12}$
Subt. ① from ②. $99N = 12$
$N = \frac{12}{99}$ or $\frac{4}{33}$

Example 3. Let ① $N = 1.231\overline{231}$
Mult. ① X 1000 ② $1000\ N = 1231.\overline{231}$
Subt. ① from ② $999N = 1230$
$N = \frac{1230}{999}$ or $\frac{410}{333}$

Example 4. Let ① $N = 1.72\overline{2}$
Multi. ① X 10 ② $10N = 17.2\overline{2}$
Multi. ① X 100 ③ $100N = 172.2\overline{2}$
Subt. ② from ③ $90N = 155.$
$N = \frac{155}{90}$ or $\frac{31}{18}$

Page	Line	Change
page 249	Problem 7(f)	$.333^{\cdots} \longrightarrow .333\ldots$
	Example 1.	$.7183946^{\cdots} \longrightarrow .7183946\ldots$ $.7184623^{\cdots} \longrightarrow .7184623\ldots$
page 250	line 5 from bottom	$x = .a_1 a_2 a_3 a_4^{\cdots} \longrightarrow .a_1 a_2 a_3 a_4\ldots$ $y = .b_1 b_2 b_3 b_4^{\cdots} \longrightarrow .b_1 b_2 b_3 b_4\ldots$
page 251	line 15.	$c = \{1, 1.7, 1.73, 1.732, 1.7320, \ldots\}$
	line 19	$1.7320^{\cdots} \longrightarrow 1.7320\ldots$
	line 20	$1.7320^{\cdots} \longrightarrow 1.7320\ldots$
page 252	line 13	Should read: For example, the unique positive solution to the equation $x^3 = 4$, $x = \sqrt[3]{4}$ is in $\mathbf{R}$; the unique positive solution to the equation $x^7 = 10$, $x = \sqrt[7]{10}$ is in $\mathbf{R}$.
page 257	line 15	$ax \longrightarrow a \cdot x$
	line 18	$= (7 + (-4)\sqrt{12} \longrightarrow (7 + (-4))\sqrt{12}$
page 259	Problem 1.	Re letter a ⟶ j
page 260	Problem 13	Should read: If a, b are real numbers $a \geq 0$, $b > 0$ then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Chapter 6

Section 6.7	p. 286 No. 4 c) $M-3.55$ d) $D\sqrt{2}-2$ No. 6 e) $\{x: -2 \leq x \leq 2\}$
Section 6.10	p. 294 Example 2: positive x-axis = $\overrightarrow{OI}$ without point O.
Section 6.15	p. 308 No. 6 (line 6 should read...) $\frac{y-3}{x-2} = \frac{7-3}{4-2}$
Section 6.17	p. 313 No. 7 G divides AF, from A to F, in the ratio 2:1. Simply AG:GF=2:1
Section 6.18	p. 317 No. 3 a) A(4,0), B(0,3) b) A(-4,0), B(0,3) c) A(5,0), B(0,12)
Section 6.23	p. 327 No. 4 Omit last sentence

PART II

Chapter 7

p. 5 - line 9 should read $1.35 \xrightarrow{d} 2.70$

p. 25- problem 6, line 3 $p, (\frac{1}{2}) = 1$

p. 44- last line any $t \in B$, $t = f(a)$ for some $a \in A$.

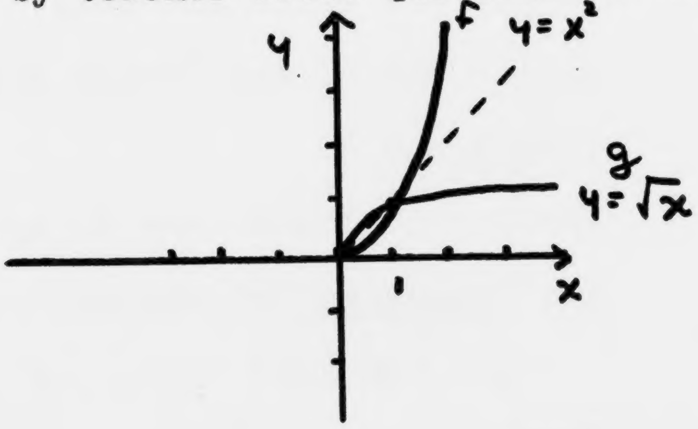
p. 48- problems 7) graphs → graph delete 3, 5.

p. 56- problem 2d change j to k.

p. 57- problem 3   Second map k should be g.

p. 58- problem 6 $x \xrightarrow{t} 3$ instead of z

p. 70- To be noted by teacher after question at end of page 70.



On page 70 of the text the above graph is what is required in the last paragraph i.e. the function f: $x \rightarrow x^2$ and its inverse g: $x \rightarrow \sqrt{x}$ for the restricted domain R_0^+.

p. 71- Figure 7.21 The teacher should indicate to the class that the unit on the x axis is not equal to the unit on the y axis and then a projection to the function curve will then give the unit measurement for the x axis and hence $\sqrt[3]{2} \approx 1.3$

p. 73 - example 4. as given "or"

$$\frac{\sqrt{2}}{\sqrt{5}}. = \frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{10}}{5}$$

p. 77 - (6) $(F, +, \cdot\ )$ is not a [field because $(F \setminus \{c\}\ \cdot\ )$ is not a group.]

p. 79 - problem 6. (m) should read $[n \cdot g](10)$

Chapter 8

page 87	5th line from last "which are which are less" strike one of the "which are's"
page 81	Line 3 and 4 The World Almanac and daily newspapers are "*examples*" of statistical ... add the word "*examples*"
page 94	Paragraph 2, second line "attention" misspelled. - atention
page 98	Exercise 1 (b) last word misspelled "lenghts" should be "lengths"
page 107	last line: "total of the frequencies" change to "*sum* of the frequencies"
page 113	Exercise 6. *Point of confusion* "Find the mean number of students per class, and compare it with the median *number of students*." Suggest dropping "number of students" (See page 121 Exercise 2a) Wording is better)

Chapter 9

p. 135	line 9	"$\overline{OP'} = 2\overline{OP}$" *not "OP' = 2OP"*
p. 152	line 7	"*r*(0, 80)" not "*R*(0, 80)"
p. 153	line 4	"*r*(0, 80)" not "*R*(0, 80)"

p. 161 Number 12 Change "Prove ts $H_F \circ H_B \circ H_F \circ H_A = i$.

p. 161 Problem 12 Change the order of compositon ts $H_F \circ H_B \circ H_F \circ H_A$.

p. 161 In the lemma "For any point X, if $X \xrightarrow{F} X'$, then $X \xrightarrow{G} X'$". should read "For any point X, if $X \xrightarrow{F} X'$ and if $X \xrightarrow{G} X'$, then F = G.

p. 169 Number 10 should be starred

p. 171 Exercise 1 After "Interpret" - add the following: "Using these results, show that the set of translations (in a plane) is a group under composition."

p. 172 Exercise 9 (c) Should read "Let R_2 be..."

p. 176 Figure at the top should be labeled "Figure 9.17"

p. 178 lines 8-9 should read $\overline{OA'}$ is $2\overline{OA}$....and $\overline{OB'} = 2\overline{OB}$

p. 183 Problem number 4 Should read "Given ABCD is a parallel- and given a point P not on the parallelogram"

p. 183 Problem number 4 Should read "Show $H_B \circ H_A$

Chapter 10

p. 212 Example 4(d) The diagonal should measure 25.

p. 222 (i) Exclude phonograph record in paragraph 1.

(ii) The rectangle in paragraph 3 should read ABJD not ABCD.

p. 223 Exercise 4 Include words in terms of π.

Chapter 1

Mathematical Logic and Proof

(Approximate Time Estimate - 14 days)

General Introduction

When teaching this chapter the teacher should be aware of the role of logic in the SSMCIS program as a whole and in Course II in particular. In the SSMCIS program, logic is seen as a tool to be utilized by the student in constructing a proof, or perhaps more correctly, as a guide for checking the mathematical correctness of his own reasoning. In Course II, this chapter is intended to help ease the transition from the informal or pre-mathematical reasoning of Course I to the formal proofs of Course II (as for example, in the Groups and Affine Geometry chapters). Therefore, this chapter should not be seen as a first course in formal mathematical logic. The material chosen for inclusion in the Chapter was deemed to be the minimum necessary for the student's later work in the SSMCIS program, so that the teacher need not be concerned with the fact that from an advanced point of view there are certain gaps and omissions. In most cases, it would be inadvisable for the teacher to try to fill the gaps or to add more advanced material, for to do so would greatly lengthen the time spent on this chapter at the expense of other material. Furthermore, full understanding and appreciation of the logic used in mathematics requires a degree of maturity and experience that few students at this level can be expected to have. We strongly recommend, therefore, that teachers try this chapter as it is written.

As illustrative examples, and in the exercises, material from Course I has been freely used, both for purposes of review and to bring to each new idea some previous mathematical experience of the student. While the review aspect of such problems provides excellent opportunities for the teacher, it should not be over-emphasized. Since the entire program is organized in a spiral manner, all the major topics of Course I will be reviewed and extended in later chapters in Course II, so that if a student is blank on some fact or area mentioned in an exercise, it is usually wise to pass on quickly after a mimimum amount of comment

Examples and exercises drawn from non-mathematical experience have been largely (though not entirely) avoided. We have found that examples of this type (used more often in an earlier version of this chapter) sometimes confuse more than clarify. The reason is that too much is connotated by such an example, so that conclusions may be drawn that have no relation to what was intended. It is possible that a student will, for some reason, encounter serious difficulties with some particular example in the text--difficulties not directly related to the bit of logic or language involved. In such cases it is usually best to move on with a remark such as "Perhaps this example is confusing you, so let's try another." In this way the teacher can usually determine whether the student's difficulty is related to the logic.

1.1 Introduction and 1.2 Mathematical Statements
(Time estimate including 1.3 = 2 days)

Several important ideas are presented in these sections: the necessity for precision in mathematical language, as opposed to ordinary language; the true or false nature of mathematical statements; and the negation of mathematical statements. The teacher should particularly reinforce the student's understanding that these concepts are meant to apply primarily to mathematical sentences.

In connection with sentences and statements, the concept of an open sentence is introduced briefly. Open sentences will be explored in greater depth in the following and later sections, so that the teacher need not dwell on them here. However, the student should be able to recognize an open sentence.

In this and later sections, truth tables are introduced, not as a means of defining connectives, but as a way of summarizing the work already done. Some work with truth tables is in the exercises; however, they are not intended to be a foundation-stone for the chapter, as they might be in a course on formal logic. It is important not to allow the truth table for a connective to obscure the meaning of the definition or the rationale behind the definition.

These sections, and the text section (1.4) that follows, are not of more than average difficulty.

1.3 Exercises

The first 11 exercises are designed to give the student experience in determining what is and what is not a statement. Exercises 12 through 21 provide work with negations of statement and should lead the student to discover a fundamental property of the negation: For any statement S, "not (not (S))" is the same as S Exercise 22 is an extension of this principle.

Some of these exercises (12-22) may be omitted. However, as a minimum, 16, 17, 18, and 21 ought to be done.

1.3 Exercises

1. Is a statement; false.
2. Is; true.
3. Not a statement, since it is a command and cannot be judged true or false.
4. Not a statement; truth value depends on replacement for x. (This is an open sentence.)
5. Is; false.
6. Is; true.
7. Is; false.
8. Is; false.
9. Is; false.
10. Is; true.
11. Is; true.
12. 721 is not prime. Original is false.
13. $71 \times 27 \neq 1917$. Original is true.

14. $\frac{1}{3} + \frac{2}{5} \geq \frac{3}{5}$: Original is true.

15. 71 is not less than 38 + 35. Original is true.

16. 1001 is not divisible by 13. Original true.

17. 1001 is divisible by 13. Original false.

18. 1001 is not divisible by 13.

19. (a) $7 \times 3 \neq 3$ in Z_9

    (b) $7 \times 3 = 3$ in Z_9

    (c) $7 \times 3 \neq 3$ in Z_9.

20. (a) 29 is prime.

    (b) 29 is not prime

    (c) 29 is prime

21 (a) S

    (b) not S

22 Not Q. General rule: If n is even same as Q; if n odd, same as not Q.

1.4 Connectives: And, Or

Time estimate (including 1.5) = 1 $\frac{1}{2}$ days

While this is an important section, most students should have little difficulty with the main ideas--definitions of the "and" and "or" compound statement. The definition, given for the "and" compound statement is a reasonable one in that it corresponds with the experience of everyday language. Some students may question the reasonableness of the definition for the "or" compound statement, since in everyday usage, "or" is most often used in an exclusive (disjunctive) way. Probably no explanation will be completely satisfying; perhaps the best the teacher can do is to indicate that the definition given is the one that has been found to be most useful mathematically. A similar difficulty comes in Section 1.6 when the conditional statement is introduced.

Finding solution sets for open compound sentences should not be considered as important as the first part of this section, so if a few students have difficulty at this point, the teacher should not dwell upon it. Likewise, the teacher need not expect all students to understand the discussion of the negation of compound statements at the end of the section. Some further work with these negations comes in the exercises.

1.5 Exercises

The first ten exercises giving practice in determining truth values of compound statements are most important, although better students will not need to do all of them. Exercises 11 and 12 throw additional light on the relationship between the two types of compound sentences introduced in this section, and should be done by all students. Exercises 13 and 14 give some practice in using truth tables and allow the students to make some discoveries about the negation of compound statements. One or both of these should be done and discussed in class.

Exercises 15-18 are discovery exercises dealing with the solution sets of compound open sentences, and if the class is having difficulty with the section in general, these exercises might be considered optional or for the better students

Answers to exercises

1. True
2. False
3. False
4. True
5. False
6. False
7 False
8. False
9. True
10. True
11. "S and T" must also be false if "S or T" is false; no conclusion can be drawn about "S and T" if "S or T" is true.
12. "G or H" must also be true if "G and H" is true; no conclusion can be drawn about "G or H" if "G and H" is false.
13.

P	Q	not P	Not Q	P or Q	(not P) and (not Q)
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	T	F
F	*F	T	T	F	T

*Note the error in the book.

Relationship between last two columns: always opposite in truth value. Conclusion: "P or Q" and "(not P) and (not Q)" are always opposite in truth value, no matter what the truth value of P and Q are, so must be negations of each other.

14.

P	Q	not P	not Q	P and Q	(not P) or (not Q)
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	F	T

The last two columns are opposite in truth value, so (as in Exercise (3) must be negations of each other.

15. Solution set for P: {6, 7, 8, 9, .. }

Solution set for Q: {..., 4,5 , 6, 7, 8}

Solution set for "P and Q": {6, 7, 8}

Solution set for "P and Q" is the intersection of the solution set for P with the solution set for Q

16. $A \cap B$

17. Solution set for V: {4, 5, 6}

Solution set for W: {6, 7, 8, 9}

Solution set for "V or W": {4, 5, 6, 7, 8, 9}

The solution set for "V or W" is the union of the solution set for V with the solution set for W.

18. $C \cup D$.

1.6 Conditional and Bi-conditional Statements (Time estimate including 1.7 = 2 days)

The definition of the conditional statement often proves to be difficult for students at first, from experience it may not seem reasonable that the conditional is true when both an antecedent and consequent are false or when the antecedent is false and the consequent is true. An effort has been made in the text to smooth over these difficulties by preparing the way for the definition with two carefully chosen examples--one mathematical and one non-mathematical--which lend reasonableness to the definition of the conditional statement. When students experience difficulty with this definition, the following approach (similar to that used in the two examples) might be used. First, the teacher can ask the student when a (given) conditional statement can be definitely said to be false. The student should realize that this can only happen when the antecedent is true and the consequent is false. Then, since the definition is to operate in a mathematical context, and the conditional statement is to be a mathematical one, in all other cases it must be true. Unfortunately, this sort of argument is not always satisfying. Students will recognize, as their experience with mathematics grows, that this definition is used because it is the most fruitful and reasonable one mathematically.

The bi-conditional statement is important because it leads to the notion of equivalence of statements. Here again, students will most likely recognize only gradually the power of equivalent statements. Note that using equivalent statements in a proof requires that the atomic statements (i. e. the smallest parts) of the equivalent complex statements be the same. For example, a truth table will show that the statements "A and not A" and "P and not P" are equivalent, but this fact is not at all useful in a proof. The teacher may wish to point this out after the exercises have been completed.

1.7 Exercises

Exercises 1-7, 9, 10 and 13 will give students practice in determining the truth value of conditional statements, while exercises 8, 11, and 12 deal with bi-conditional statements. Note that of these, only one (Exercise 5) is non-mathematical. Some students may argue that because this statement is non-mathematical the definition cannot be applied and no conclusion can be drawn about the truth or falsity of the statement. This is certainly a valid argument and can help the students discover the important distinction between mathematical and non-mathematical statements and arguments. Most students should do all of these introductory exercises, some or all of Exercises 14-16 (finding negations of conditional and bi-conditional statements), and Exercise 17, as a minimum. Exercise 18 is designed to illuminate an important point about symmetry and transitivity of relations, using the logic developed in this section (see answer to Exercise 18 below). Exercises 19 and 20 explore further equivalent statements from the point of view of truth tables. Exercise 20 is more difficult than the others and may be left to the better students.

Answers to Exercises

1. True
2. False
3. True
4. True
5. True (if we apply the definition and regard the antecedent as false)
6. True
7. False
8. False
9. True
10. True
11. True
12. True
13. True
14. 6 is odd and 3 x 6 is even.
15. (a) 3 is even and 5 is odd, and 3 + 5 is odd.
    (b) 3 + 5 is even, and 3 is odd or 5 is even.
    (c) 3 is even and 5 is odd, and 3 + 5 is odd, or:
    3 + 5 is even, and 3 is odd, or 5 is even.
16. (a) 4 is odd and 3 x 4 is even.
    (b) 3 x 4 is odd and 4 is even.
    (c) 4 is odd and 3 x 4 is even, or:
    3 x 4 is odd and 4 is even.

Note: The bi-conditional "A iff B" is equivalent to the compound statement "(if A, then B) and (if B, then A)."

To negate this, we have "not (if A, then B) and (if B, then A) " which, from Section 1.4, is equivalent to: "not (if A, then B) or not (if B, then A)". Now, since "not (if A then B)" is equivalent to "A and not B," and "not (if B, then A)" is equivalent to "B and not A", we have finally "A and not B" or "B and not A" as the negation of the bi-conditional "A iff B". Hopefully, students will be able to write this negation for the specific examples of these two exercises without going through this long and rather formal argument. The teacher should not spend too much time on these two exercises.

17.

P	Q	not Q	if P, then Q	not (if P, then Q)	P and(not Q)
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

18. R is symmetric; for, whenever xRy is true (i. e. only when x = y), then yRx is true. R is transitive; for whenever aRb and bRc are true (i. e. only when a = b = c), then aRc is true also.

19.

S	T	U	T and U	S or T	S or U	S or (T and U)	(S or T) and (S or U)
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	T	F	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

The last two columns have the same truth values at every line, so the statements "S or (T and U)" and "(S or T) and (S or U)" are equivalent.

20.

A	B	C	B or C	not C	If A, then B	If (not C), then (if A, thenB)	If A, then (B or C)
T	T	T	T	F	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	T	T
F	F	F	F	T	T	T	T

1.8 Quantified Statements (Time estimate for 1.8 - 1.10 = 1 1/2 days)

This section develops the concepts of universal and existential statements and quantifiers. Many examples are given emphasizing the different forms statements can take. Negating a quantified statement is described and it is suggested that teachers either have the students form the negation of each of the examples in this section or do some of the 1.10 exercises in class.

1.9 Substitution Principle for Equality (SPE)

This section defines SPE and shows that left operation is a result of it. The main idea of this section can be emphasized with just a few examples since it can be re-emphasized in the sections on inference and proof.

1.10 Exercises (exercises 19 and 20 might pose difficulties.)

1.	True	Universal
2.	False	Universal
3.	True	Existential
4.	True	Existential
5.	False	Universal
6.	True	Existential
7.	False	Universal
8.	False	Existential
9.	True	Existential
10.	False	Universal
11.	True	Existential
12.	False	Universal
13.	False	Universal
14.	True	Existential
15.	False	Universal

16. Negations
    1. Some line reflections are not isometries. Existential.
    2. Some isometries are not line reflections. Existential.
    3. All line reflections are not isometries. Universal.
    4. All isometries are not line reflections. Universal

17. Negations

5. For some integers x and y, x^2y^2 is odd.

6. For all integers x and y, x^2y^2 is odd.

18. Negations

10. For some mappings s and t, $s \circ t \neq t \circ s$.

11. For all mappings g and h, $g \circ h = h \circ g$.

19. True.

Negation: for some integer s, there does not exist an integer t such that $t > s$.

20. False

Negation: for all integers x, there exists a y such that $y \leq x$.

21. SPE is used for the following substitutions

$37 = 30 + 7$

$53 = 50 + 3$

$30 \times 50 = 1500$

$7 \times 50 = 350$

$30 \times 3 = 90$

$7 \times 3 = 21$

$1500 + 350 + 90 + 21 = 1961$

22. SPE is used **for** the following substitutions

$0 = 0 + 0$

$r \cdot (0 + 0) = r \cdot 0 + r \cdot 0$

1.11 Inference (Time estimate including 1.12 = 2 days)

In this section, 5 rules of inference are discussed and many examples are given. It is important for each example to point out the form of the argument and to remind the student not to base conclusions on any outside or previous knowledge but only on what is assumed true. In order to separate the role of inference from the truth value of statements, examples are used where false statements are assumed true (as in example 3d page 41). If form is considered, these examples should pose no problems. It will be helpful when going over the examples to assign letter names to the statements and to classify the form of the argument presented.

It should also be pointed out to the students that statements of the form "all A are B" can be interpreted as "if x is an A then it is a B."

Example 7 (page 44) which states "all isometries preserve angle measure" is of this form and can be interpreted as "if x is an isometry, then x preserves angle measure."

1.12 Exercises

These exercises are concerned mainly with form. It might be a good idea to have all students do Part (1) for exercises 1-15 (that is state the inferences) and to assign some problems to each student to complete part (2). Note that problem 12 is the most complicated. Question 16 should be attempted after the other questions are fully understood. This problem might also be used as a special assignment.

Answers to Exercises

1. P: 1 Toss a Fair coin
   Q: The probability of getting a tail is 1/2
   Assumed true, if P then Q, P
   Conclusion, Q
   by inference Rule (1), Rule of detachment
2. R: Set A is a subset of every set
   S: Set A is the empty set
   Assumed true, if R then S, not S
   Conclusion not R: set A is not a subset of every set
   by inference Rule (2)
3. P: Sets A and B are the complements of each other
   Q: The union of the two sets is the universe
   Assumed true, if P then Q, Q
   No conclusion
4. R: The image of point A under a reflection in point P is A'
   S: P is the midpoint of $\overline{AA'}$
   Assumed true, if R then S, R
   Conclusion, S
   by inference Rule (1)
5. A: M is parallel to N
   B: M is parallel to P
   Assumed true, A or B, A
   No conclusion
6. P: B is between A and C
   Q: AB + BC = AC

Assumed true, if P then Q, not Q
Conclusion Not P, B is not between A and C
by inference Rule (2)

7. R: The natural number 7 is even
S: The natural number 7 is odd
Assumed true R or S, not R
Conclusion, S
by inference Rule (5)

8. A: x and y are both positive
B: The product of x and y is positive
Assumed true, if A then B, B
No conclusion

9. P: The sum of a and b is negative
Q: At least one of a, b is negative
Assumed true, if P then Q, not P
No conclusion

10. R: x and y are both positive
S: The product of x and y is positive
Assumed true if R then S, not R
No conclusion

11. P: a and b are rational numbers
Q: There is a rational number between a and b
R: a and b are greater than five.
Assumed true if P then Q, P and R
Conclusions, P from P and R by
inference rule (3)
then Q by inference rule (1)

12. This example contains more simple statements than first meet the eye. One way to attack this problem is as follows:

$x \geq 8$ is equivalent to $x > 8$ or $x = 8$

$|x - 5| > 3$ is equivalent to $x > 8$ or $x < 2$

and $y \geq 5$ is equivalent to Not $y < 5$

therefore assign letters as follows:

A: $x > 8$

B: $x = 8$

C: $x < 2$

D: $y < 5$

Our original statements which we assume true then become:

"if (A or B) then Not D",

"(A or C) and D"

from the second statement we can infer

D using inference Rule (3)

D is equivalent to not (not D)

Thus we can infer not (A or B)

from the first statement using inference Rule (1)

Not (A or B) is equivalent to "Not A and Not B" from

this we can infer Not A using inference Rule (3)

From the second statement we can also infer A or C

using inference Rule (3)

Finally from the last two conclusions we can infer C

using inference Rule (5)

Thus our conclusion is $x < 2$.

13. P: a number is divisible by 8

Q: a number is divisible by 4

R: a number is divisible by 2

Note the use of the universal statements.

In this case the substitution instance is 88 for the term "a number"

Assumed true: if P then Q, if Q then R
and P.

Conclusion Q by inference rule (1)

then R also by inference rule (1)

Thus the specific conclusion reads 88 is divisible by 2.

14. A: $x = 3$

B: $x = 4$

C: $y = 7$

Assumed true A or B, if A then C, not C

Conclusions first not A using the last two statements and inference Rule (2)

Then B using inference Rule (5)

15. P: $\overrightarrow{AC} = \overrightarrow{AB}$

Q: B is on $\overrightarrow{AC}$

R: $\overrightarrow{AC} \cap \overrightarrow{AB} = A$

Assumed true, if P then Q
P or R
Not Q

Conclusions Not P using inference Rule (1)

Then R using inference Rule (5)

16. The murderer was the stepson.

Using letter names the argument becomes:

A: The butler murdered Mr. X

B: The stepson murdered Mr. X

C: The murder occurred before midnight

D: The stepson's testimony is correct

E: The house lights were turned off at midnight

F: The butler is wealthy

1. A or B
2. If A then not C
3. If D then C
4. If not D then not E
5. E and not F

From 5. infer E	Rule (3)
From E and 4. infer D	Rule (2)
From D and 3. infer C	Rule (1)
From C and 2. infer not A	Rule (2)
From not A and 1. infer B	Rule (5)

1.13 Direct Mathematical Proof and 1.14 Indirect Mathematical Proof (Time estimate including 1.15 = 3 days)

These two sections give detailed and rigorous examples of direct and indirect mathematical proofs emphasizing three strategies. The emphasis here should be on the strategies used not necessarily on the rigor since, as mentioned in the text, abbreviated forms of proof are usually given and the degree of abbreviation will depend to a great extent on the individual teacher and his students. A careful analysis is given of the proof of the statement "if a and b are even whole numbers, then a + b is even." Students should be reminded that although they may give abbreviated forms of mathematical proofs, a similar analysis of all steps should be possible and they should be able to justify all conclusions.

1.15 Exercises

1. Theorem C. If _a_ or _b_ is even,
   then a·b is even

   Proof (Given on pages 52-53)

   Analysis:

   Step 1. The strategy here is to assume the antecedent true and to show the consequent follows also as true.

   Step 2. This is an argument by cases that is for "a or b is even" to be true "a is even" or "b is even" or "both a and b are even" will be true. We take the first case as true and claim a similar argument will hold for the other cases.

   Step 3. Using the definition, _N_ is even if and only if N = 2M for some _M_ in W, and the assumption _a_ is even from Step 2 we conclude a = 2x by inference Rule (1)

   Step 4. Right operation principle states that for any _x_, _y_, _z_ in S, if x = y then x o z = y o z in (S, o). In particular using a = 2x then a·b = (2x)·b in (W,·)

Step 5. The associativity property of multiplication in $(W, \cdot)$ states, if *x*, *y*, *z* are elements of W, then $(xy)z = x(yz)$. Since 2, x and b are elements of W, inference Rule (1) justifies the statement $(2x)b = 2(xb)$.

Step 6. Using the statements of 4 and 5, SPE justifies stating $a \cdot b = 2(x \cdot b)$

Step 7. The closure property and inference Rule (1) justify this statement.

Step 8. Inference Rule (1) and the definition given above justify this statement

2. Theorem: Let *a*, *b*, and *c* be whole numbers. If *a* divides *b* and *b* divides *c*, then *a* divides *c*.

Proof: A direct strategy is used.

Steps	Reasons
1. *a* divides *b* and *b* divides *c*.	1. Assumption
2. *a* divides *b*.	2. Inference Rule (3)
3. $b = ax$ for some *x* in W.	3. Definition of divides
4. *b* divides *c*.	4. Inference Rule (3) on step 1
5. $c = by$ for some *y* in W.	5. Definition of divides
6. $c = (ax)y$.	6. SPE step 3 in step 5
7. $(ax)y = a(xy)$.	7. Associativity of Mult.
8. $c = a(xy)$.	8. SPE step 6 in step 7
9. xy is in W.	9. Closure property
10. *a* divides *c*.	10. Definition of divides

3. Assertion: If a + b is odd and *a* is even
Then *b* is odd.

Proof: An indirect approach is used.
We will assume "P and (not Q)" and
show a contradiction.

1. a + b is odd and *a* is even and *b* is even.	1. Assumption
2. a + b is odd. *a* is even. *b* is even	2. Inference Rule 3 used twice
3. *a* and *b* are even	3. by inference Rule 4
4. a + b is even	4. Theorem A
5. a + b is both even and odd.	5. Steps 2 and 4

Therefore "P and (not Q)" has led to a false statement and must itself be false.

4. Assertion: If a + b is even, then *a* is even and *b* is even.

Counter example:

5 + 3 is even but 5 is *not* even and 3 is *not* even.

This assertion is not true. The one counterexample above disproves it. This exercise was included to point out to students that because something is *asserted* does not mean it is true.* This exercise can also be used to point out to students that only one counterexample is needed to disprove an assertion.

*It will be interesting to see just what "proofs" if any are given.

5. Assertion: If a, b and c are odd,
Then ab + ac is even

Proof: A direct strategy is used.

1. a, b, and c are odd	1. Assumption
2. a is odd.	2. Inference rule (3)
b is odd.	3. Step 2 and
c is odd.	Inference rule (4)
3. b and c are odd	4. Theorem B
4. b + c is even	5. Theorem C
5. a ·(b+c) is even	6. Distributive property
6. a ·(b+c) = ab + ac	of (W, T ·)
7. ab + ac is even	7. SPE step 6 in step 5

1.16 Summary (Time estimate including 1.17 = 1 day)

1.17 Review Exercises

1. a. True statement
   b. True statement
   c. Not a statement - a command.
   d. False statement
   e. False statement
2. The compound statements are easily formed by inserting "and" or "or" between the two simple statements.
   The negations are formed by changing
   a) "are" to "are not", and
   "is" to "is not".

b) $=$ to $\neq$, and

"Every prime number has" to "Some prime numbers do not have"

c) $>$ to $\leq$, and

"Some triangles have" to "Every triangle does not have."

d) "is" to "is not", and

"All cats have" to "Some cats do not have."

The negations of the compounds are easily formed if we consider "Not (P or Q)" equivalent to "Not P and not Q" and "Not (P and Q)" equivalent to "Not P or not Q."

3. (a) if $a > b$ then $a + c > b + c$.

   True

   (b) if $x \neq 0$ then $x^2 > 0$.

   True

   (c) if $x/x \neq 1$ then $x = 0$

   True

4. (a) I do not go swimming

   (b) the water is not cold

   (c) No conclusion

   (d) No conclusion

   (e) The Yankees did not win the pennant.

   The Yankees did not play well.

   (f) $(742)^2$ is an integer

Chapter 1: Suggested Test Items

I. Given the following 3 statements:

A: 323 is prime.

B: 323 is a multiple of 17.

C: 17 is even.

Where A and C are false and B is true,

a) Write out the following statements and give their truth values

1. A and B:
2. If C then B:
3. A or C:
4. Not A:
5. If not A then C:
6. Not B and not C:
7. If B then not A:

b) Using the simple statements A, B, and C form an example of each of the following compound statements. (use examples different from those given in part a).)

1. a false compound and statement
2. a true compound or statement
3. a false conditional statement
4. a true biconditional statement

II. In each of the following, assume the given statements are true and determine what inferences you can make.

a) If N is a natural number, then 2·N is even. 5 is a natural number.

b) x is an even number or x is a perfect square. x is not an even number.

c) If x is even then x is not a perfect square. x is a perfect square.

d) $a \geq b$ and $b < c$. If $b < c$ then $a = d$.

e) If 8 is a factor of x then 4 is a factor of x.
9 is a factor of x or 8 is a factor of x.
4 is not a factor of x.

III. Explain what is meant in mathematics by the term proof. Include in your discussion the differences between direct and indirect proofs and describe if possible strategies used in proving mathematical statements.

Answers to Suggested Test Items

I.a) 1. 323 is prime and 323 is a multiple of 17. False
2. If 17 is even then 323 is a multiple of 17. True.
3. 323 is prime or 17 is even. False.
4. 323 is not prime. True
5. If 323 is not prime then 17 is even. False.
6. 323 is not a multiple of 17 and 17 is not even. False.
7. If 323 is a multiple of 17 then 323 is not prime. True"

b) 1. "A and B"
"A and C"
or "B and C"

2. "A or B"
   "B or C"
3. "if B then A"
   " if B then C"
4. "A iff C"

II. a) $2 \cdot 5$ is even
   b) x is a perfect square
   c) x is not even
   d) $a = d$
   e) 9 is a factor of x

III. The main points which the students should include are the following:

A mathematical statement to be proven is usually in the form of a conditional. A proof consists of a sequence of statements leading to the desired conclusion. Each step is justifiable as an axiom, definition or theorem, or as the result of an inference from previous statements. A direct proof begins by assuming P true and showing Q will follow as true also. An indirect proof begins by assuming Q false and showing that P would follow also as false or by assuming "P and (not Q)" and showing that this leads to a contradiction.

Chapter 2

Groups

Time Estimate for Chapter: 16 - 17 days

The principal objectives of this chapter are to:

1. become aware of the great prevalence of groups in mathematics,
2. learn the definition of a group,
3. appreciate the unifying power of group theorems,
4. be exposed to another axiomatic system,
5. be exposed to additional proofs at a more formal level,
6. learn of permutations and permutation groups,
7. learn functional notation,
8. learn the meaning of a^n for an operational system,
9. learn a few basic group theorems and their proofs,
10. learn what an isomorphism is and what isomorphic groups are.

2.1 Definition of a Group (Time: 3 days)

In 2.1 an effort is made to convey the importance of groups by showing how prevalent they are and how they might serve to unify apparently diverse situations in mathematics. Many illustrations of groups are provided before arriving at a definition of a group.

Time should be taken with this initial section because of its great importance in developing the definitions and properties of fields and rings which occur in subsequent chapters and courses. The program stresses the use of groups in explaining the step by step solution to linear equations.

2.2 Exercises

1. (a) 1 (b) 0 (c) e

2. (a) -5 (b) $\frac{1}{5}$ (c) 1 (d) x (e) 3

3. (a) No identity, not associative
   (b) No operational system
   (c) 0 has no inverse
   (d) No identity element
   (e) 2 has no inverse
   (f) No identity element
   (g) No identity element
   (h) No operational system
   (i) No element has an inverse except 1.
   (j) No identity element
   (k) No identity element
   (l) No operational system
   (m) No identity element
   (n) 0 and 2 do not have inverses

4.

○	S	L	A	R
S	S	L	A	R
L	L	A	R	S
A	A	R	S	L
R	R	S	L	A

(a) {S,L,A,R} a set of commands

(b) followed by assigns to every pair of commands a command.

(c) Stay or S

(d) Yes

(e) (1) - (4) All Yes

(f) Yes. Associativity, unless each command is regarded as a mapping.

5.

$\circ$	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

(a) (0,0)

(b) (0,1)

(c) $[(0,1) \circ (1,1)] \circ (1,0) = (1,0) \circ (1,0) = (0,0)$

$(0,1) \circ [(1,1) \circ (1,0)] = (0,1) \circ (0,1) = (0,0)$

(d) Yes

6. (a) Yes (b) No, not associative and no identity.

7. (a) Yes (b) Yes (c) Yes (d) 1,4

8. (a) $(Z,\cdot)$ and others.

(b)

$\circ$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

Not associative as $(1\circ1)\circ2 = 0\circ2 = 2$ and $1\circ(1\circ2) = 1\circ1 = 0$

0 is the identity element. Each element is its own inverse.

(c) $(Z,\ a\circ b =$ larger of $\{a,b\})$

(d)

o	0	1	2
0	0	1	2
1	1	0	1
2	2	1	2

Not associative as $(1 o 1) o 2 = 0 o 2 = 2$

and $1 o (1 o 2) = 1 o 1 = 0$

Identity is 0

2 has no inverse

9. (a) No; there is no identity

(b) Yes

(c)

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

$(Z_7 \setminus \{0\}, \cdot)$ is a group.

$(\{1,6\}, \cdot)$ is a subgroup, and there are others.

10. No; there is no identity

11. (a) Yes (b) Yes (c) No

12. Yes; $na + nb = n(a + b)$

• 13. $(T,\circ)$ is associative since $(S,\circ)$ is. If $x \in T$, then $x^I \in T$. Therefore $x \circ x^I = e$, $e \in T$.

14. (a)

Δ	{ }	{a}	{b}	{a,b}
{ }	{ }	{a}	{b}	{a,b}
{a}	{a}	{ }	{a,b}	{b}
{b}	{b}	{a,b}	{ }	{a}
{a,b}	{a,b}	{b}	{a}	{ }

(b) $\{\ \}$ is the identity element

(c) $\{b\}^I = \{b\}$

(d) $(\{a\} \Delta \{b\}) \Delta \{b\} = \{a,b\} \Delta \{b\} = \{a\}$

$\{a\} \Delta (\{b\} \Delta \{b\}) = \{a\} \Delta \{\ \} = \{a\}$

(e) Yes. It possesses the required properties.

15.

Δ	{ }	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
{ }	{ }	{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}
{a}	{a}	{ }	{a,b}	{a,c}	{b}	{c}	{a,b,c}	{b,c}
{b}	{b}	{a,b}	{ }	{b,c}	{a}	{a,b,c}	{c}	{a,c}
{c}	{c}	{a,c}	{b,c}	{ }	{a,b,c}	{a}	{b}	{a,b}
{a,b}	{a,b}	{b}	{a}	{a,b,c}	{ }	{b,c}	{a,c}	{c}
{a,c}	{a,c}	{c}	{a,b,c}	{a}	{b,c}	{ }	{a,b}	{b}
{b,c}	{b,c}	{a,b,c}	{c}	{b}	{a,c}	{a,b}	{ }	{a}
{a,b,c}	{a,b,c}	{b,c}	{a,c}	{a,b}	{c}	{b}	{a}	{ }

(b), (c), (d) same as exercise 11

16. (a) (0,0), (0,1), (0,2), (1,0), (1,1), (1,2)

(b)

+	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,1)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(0,2)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)
(1,0)	(1,0)	(1,1)	(1,2)	(0,0)	(0,1)	(0,2)
(1,1)	(1,1)	(1,2)	(1,0)	(0,1)	(0,2)	(0,0)
(1,2)	(1,2)	(1,0)	(1,1)	(0,2)	(0,0)	(0,1)

Identity element is (0,0)

17. (a) (1) $(Z,+)$ is an operational system with addition being associative. The identity element is 0. The inverse of any element a is its additive inverse $-a$.

(2) $(Q \setminus \{0\},\cdot)$ is an operational system with multiplication being associative. The identity is 1, the inverse of every element $\frac{a}{b}$ is its reciprocal $\frac{b}{a}$.

(b) (1) $(Z \setminus \{0\},+)$ is not an operational system as $-1 + 1$ is not in the set $Z \setminus \{0\}$.

(2) $(Q,\cdot)$ is not a group because 0 does not have an inverse.

18. (a) $2^a \cdot 2^b = 2^{a+b}$ is defined for all a, b in Z.

(b) $(2^a \cdot 2^b) \cdot 2^c = (2^{a+b}) \cdot 2^c = 2^{(a+b)+c} = 2^{a+(b+c)}$

$2^a \cdot (2^b \cdot 2^c) = 2^a \cdot (2^{b+c}) = 2^{a+(b+c)}$

Hence we have associativity.

2^0 is the identity element as $2^0 \cdot 2^a = 2^{0+a} = 2^a$

$2^a \cdot 2^0 = 2^{a+0} = 2^a$

The inverse of 2^a is 2^{-a} as $2^a \cdot 2^{-a} = 2^{a+-a} = 2^0$.

(c) The identity is 2^0 or 1.

$(2^4)^I = 2^{-4}$; $(2^{-3})^I = 2^3$; $(2^1)^I = 2^{-1}$; $(2^0)^I = 2^0$

19. In $(Z_n,\cdot)$, 1 is the identity. 0 can have no inverse since there is no element *a* such that $0 \cdot a = 1$.

* 20. If $n = 2$, the set is $\{1\}$, and there *is* a group of one element. Suppose, however $n>2$ and n is even. Then $\frac{n}{2} = k$ is in the set. Thus, 2 and k are in the set. However,

$2 \cdot k = 0$ is not in the set.

* 21. p and q are both less than n and hence are in Z_n. However, $p \cdot q = 0$ is not in Z_n.

2.3 A non-Commutative Group

In 2.3 the student is exposed, for the first time, to a non-commutative group. Since all previous groups discussed were commutative, the student often draws the conclusion that all groups are commutative. This vivid example of a non-commutative group is intended to steer the students' thinking back on those properties that are essential for a group as opposed to those that are not essential.

2.4 Exercises

Exercise 1 is a good illustrative example and may be done with the students in class. The remaining exercises can be assigned for homework.

1. (a) e: abc
p: bca
q: cab
r: acb
s: cba
t: bac

(b) e and r do not alter the position of a
e and s do not alter the position of b
e and t do not alter the position of c

(c) (1) {t}
(2) {s}
(3) {s}
(4) {r}
(5) {e,r,s,t}
(6) {q}

(7) { }

(8) {s}

(9) {p}

(10) {s}

(11) {e,p,q}

(12) {e,r}

(13) {s}

(14) {t}

(15) {e}

(16) {e,p,q,r}

(17) {r,s,t}

(18) {s}

(19) {e}

(20) {e,p,q}

(d) (1) $(poq)^I = e$, $p^I oq^I = e$, $q^I op^I = e$. All are e.

(2) $(por)^I = s$, $p^I or^I = t$, $r^I op^I = s$, $(por)^I = r^I op^I$

(3) $(qot)^I = s$, $q^I ot^I = r$, $t^I oq^I = s$, $(qot)^I = t^I oq^I$

(4) $(xoy)^I = y^I ox^I$ is the conjecture.

(5) $(p^I)^I = p$, $(q^I)^I = q$, $(r^I)^I = r$, $(x^I)^I = x$ is the conjecture.

(e) ({e,p,q}, o) is an operational system for:

o	e	p	q
e	e	p	q
p	p	q	e
q	q	e	p

The identity element is e.

Associativity follows from the fact that ({e,p,q,r,s,t},o) is a group.

({e,p,q},o) has the inverse property for $p^I = q$, $q^I = p$, $e^I = e$.

(f) The subgroups of ({e,p,q,r,s,t},o) are: ({e},o), ({e,p,q},o), ({e,r},o), ({e,s},O), ({e,t},o),

$(\{e,p,q,r,s,t\},o)$.

(g) (1) $\{e,r\}$, $\{qoe\}$, $\{qor\} = \{q,t\}$, $\{soe,sor\} = \{s,p\}$,

$\{e,p,q,r,s,t\} = \{e,r\} \cup \{q,t\} \cup \{s,p\}$.

(2) $\{e,r\}$, $\{eop,rop\} = \{p,t\}$, $\{eoq,roq\} = \{q,s\}$

$\{e,p,q,r,s,t\} = \{e,r\} \cup \{p,s\} \cup \{q,s\}$

(3) (a) $\{e,s\}$, $\{p,t\}$, $\{q,r\}$

or $\{e,s\}$ $\{p,r\}$, $\{q,t\}$

(b) $\{e,t\}$, $\{p,r\}$, $\{q,s\}$

or $\{e,t\}$, $\{p,s\}$, $\{q,r\}$

(c) $\{e,p,q\}$, $\{r,s,t\}$

2. (a) $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 1 & 2 & 3\end{pmatrix}$. and $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}$

(b) $\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix}$

(c) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}$

(d) $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$ and $\begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}$, yes

3. (a) $\ell_x = \begin{pmatrix}A & B & C & D\\ D & C & B & A\end{pmatrix}$ (b) $\ell_y = \begin{pmatrix}A & B & C & D\\ B & A & D & C\end{pmatrix}$

(c) $P_o = \begin{pmatrix}A & B & C & D\\ C & D & A & B\end{pmatrix}$

(d)
$$\begin{array}{c|cccc} o & e & \ell_x & \ell_y & P_o \\ \hline e & e & \ell_x & \ell_y & P_o \\ \ell_x & \ell_x & e & P_o & \ell_y \\ \ell_y & \ell_y & P_o & e & \ell_x \\ P_o & P_o & \ell_y & \ell_x & e \end{array}$$

(e) $(\{e,\ell_x,\ell_y,P_o\},o)$ is a group.

2.5 More on Permutations (Time for 2.5 - 2.8: 2 days)

In section 2.5 there is a discussion of 1-1 into and onto mappings $S \longrightarrow S$. For a finite set 1-1 into and 1-1 onto $S \longrightarrow S$ conditions for a mapping are the same, each implying the other. For infinite sets this is not the case.

2.6 Functional Notation

In section 2.6 functional notation is discussed briefly. It is important to relate functional notation to mappings.

2.7 More Notation

In 2.7 exponential notation for an operational system is discussed. If there is but one operation, then there should be no confusion. If two operations are present then multiplication is the one that is used for exponentials. Thus 3^2 in $(Z,+)$ means $3 + 3$ but in $(Z,+,\cdot)$ 3^2 means $3 \cdot 3$. However, it is better to avoid the symbol 3^2 when working with $(Z,+)$. The use of a coefficient is better. Thus, in $(Z,+)$ rather than using "a^2" use "$2a$", although according to our convention, $a^2 = a + a$ and $2a = a + a$, for $(Z,+)$.

2.8 Exercises

1. (b) (c) and (d) are 1-1, onto, and permutations. The others are none of these.
2. (a) None.

   (b) 1-1

(c) 1-1, onto, permutation.

(d) 1-1, onto, permutation.

(e) onto. Note: For every positive integer k there is a positive integer n such that

$$k = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right]$$

3. (a) There are many answers. We give but one for each mapping.

		Domain	Range
	f_1	$\{-3,-2,-1,0,1,2,3\}$	$\{0,1,4,9\}$
	f_2	Z	Even Integers
(b)	f_3	Even Integers	Z
	f_4	Even Integers	Z
	f_5	Z	$Z \cup \{\frac{1}{2}\}$

4. (a) None.
   (b) 1-1, onto, permutation.
   (c) 1-1, onto, permutation.
   (d) None.
   (e) None.

5. (a) 4/9
   (b) 56.25
   (c) 549.025
   (d) 9/64
   (e) 81
   (f) aaaa or a^4
   (g) 6.5
   (h) .65
   (i) 9
   (j) 6.25
   (k) 18
   (l) 12.5
   (m) 3
   (n) .9
   (o) $2\frac{1}{3}$
   (p) .8
   (q) 3/7
   (r) 1.25
   (s) 1/49
   (t) 1/49
   (u) 49/50 or .98

6. (a) 5 (g) 5 (m) 5/4
   (b) -4 (h) 5 (n) -4
   (c) 1/5 (i) 5 (o) 5/4
   (d) -1/4 (j) 5 (p) 4/5
   (e) 5/4 (k) 5 (q) -1/4
   (f) 4/5 (l) 1/5 (r) 4/5

7. (a)

o	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_4	f_3	f_6	f_5
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_2	f_6	f_3	f_1
f_5	f_5	f_4	f_6	f_2	f_1	f_3
f_6	f_6	f_3	f_5	f_1	f_2	f_4

$f_1: n \longrightarrow n$

$f_2: n \longrightarrow 1-n$

$f_3: n \longrightarrow \frac{1}{n}$

$f_4: n \longrightarrow \frac{1}{1-n}$

$f_5: n \longrightarrow \frac{n}{n-1}$

$f_6: n \longrightarrow \frac{n-1}{n}$

(b) If we make the following replacements the two tables will be identical except for the arrangement of rows and columns.

$f_1 \longleftrightarrow e$ $f_4 \longleftrightarrow q$

$f_2 \longleftrightarrow r$ $f_5 \longleftrightarrow t$

$f_3 \longleftrightarrow s$ $f_6 \longleftrightarrow p$

(c) Yes.

(d) $(f_3 \circ f_5)^I = f_4^I = f_6$

8. (a) 4 (d) 0
   (b) 1 (e) 3
   (c) 9 (f) 3

9. (a) $(f_4^2)(5) = f_6(5) = 4/5$ (b) same as (a)

(c) $(f_8^2)(5) = f_4(5) = -1/4$ (d) same as (c)

10. $1 = 3^0$ $\qquad 4 = 3^4$
$2 = 3^1$ $\qquad 5 = 3^5$
$3 = 3^1$ $\qquad 6 = 3^3$

$1 = 5^0$ $\qquad 4 = 5^2$
$2 = 5^4$ $\qquad 5 = 5^1$
$3 = 5^5$ $\qquad 6 = 5^3$

11. Slide Rule construction.

12. $1 = 2^0 = 6^0 = 7^0 = 8^0 = 10^0$
$2 = 2^1 = 6^9 = 7^3 = 8^7 = 10^9$
$3 = 2^8 = 6^2 = 7^4 = 8^6 = 10^8$
$4 = 2^2 = 6^4 = 7^6 = 8^4 = 10^7$
$5 = 2^4 = 6^6 = 7^2 = 8^8 = 10^6$
$6 = 2^9 = 6^1 = 7^7 = 8^3 = 10^5$
$7 = 2^7 = 6^3 = 7^1 = 8^9 = 10^4$
$8 = 2^3 = 6^7 = 7^9 = 8^1 = 10^3$
$9 = 2^6 = 6^4 = 7^8 = 8^2 = 10^2$
$10 = 2^5 = 6^5 = 7^5 = 8^5 = 10^1$

(c) (1) 5
(2) 1
(3) 10
(4) 5

Hence, we could use powers of 6, 7, 8 as well as 2 for our slide rule contstruction.

2.9 Some Theorems About Groups (Time: 5 days)

In 2.9 eight basic group theorems are proved. Take time with the proofs. Make sure that the reasons for the steps are understood. A common difficulty is to give an incorrect reason.

The position should be taken that unless a correct reason is available for a statement in a proof the statement should not be made.

There are a variety of approaches that may be followed in addition to those taken in the text. Students should be encouraged to use their ingenuity in varying the approaches and methods used in proving these group theroems.

2.10 Exercises

Exercises 5 and 7 are excellent for classroom demonstrations. They demonstrate all of the group theorems presented in section 2.9.

1.	$(Z_7, +)$	$(Z_7 \setminus \{0\}, \cdot)$
(a)	$(3^I)^I = 4^I = 3$	$(3^I)^I = 5^I = 3$
(b)	$(3 + 4)^I = 0^I = 0$ $3^I + 4^I = 4 + 3 = 0$	
(c)		$(3 \cdot 4)^I = 5^I = 3$ $4^I \cdot 3^I = 2 \cdot 5 = 3$

2.		
	$(a \circ b)^I = b^I \circ a^I$	Theorem 8
	$= a^I \circ b^I$	This group is commutative.

3. Let x and y be any pair of elements in the group. We must show that $x \circ y = y \circ x$.

$(x^I)^I = x \qquad (y^I)^I = y$	Theorem 7
$x \circ y = x \circ y$	Equality is reflexive
$x \circ y = (x^I)^I \circ (y^I)^I$	Replacement
$= (y^I \circ x^I)^I$	Theorem 8
$(y^I \circ x^I)^I = (x^I \circ y^I)^I$	Assumption

$x \circ y = (x^I \circ y^I)^I$ Equality is transitive

$= (y^I)^I \circ (x^I)^I$ Theorem 8

$= y \circ x$ Theorem 7

4. $(a \circ a^I) \circ (a^I)^I = a \circ (a^I \circ (a^I)^I)$ Associativity

$e \circ (a^I)^I = a \circ e$ Definition of Inverse

$(a^I)^I = a$ Definition of e

5. $(a \circ b)^I = (a \circ b)^I$ Equality is reflexive.

$(a \circ b)^I \circ ((a \circ b) \circ (b^I \circ a^I)) = ((a \circ b)^I \circ (a \circ b)) \circ (b^I \circ a^I)$ Associativity

$(a \circ b)^I \circ (\{(a \circ b) \circ b^I\} \circ a^I) = e \circ (b^I \circ a^I)$ Associativity and definition of inverse

$(a \circ b)^I \circ (\{a \circ (b \circ b^I)\} \circ a^I) = b^I \circ a^I$ Associativity and definition of identity

$(a \circ b)^I \circ (\{a \circ e\} \circ a^I) = b^I \circ a^I$ Definition of inverse

$(a \circ b)^I \circ (a \circ a^I) = b^I \circ a^I$ Definition of e

$(a \circ b)^I \circ e = b^I \circ a^I$ Definition of inverse

$(a \circ b)^I = b^I \circ a^I$ Definition of e

6. (a) $(p \circ r)^I = s^I = s$ and

$p^I \circ r^I = q \circ r = t$

(b) $r^I \circ p^I = r \circ q = s = (p \circ r)^I$ from (a)

(c) $(p \circ s)^I = t$, $p^I \circ s^I = r$, $s^I \circ p^I = t$ therefore

$(p \circ s)^I \neq p^I \circ s^I$, $(p \circ s)^I = s^I \circ p^I$ and many other pairs will do.

7. (a) $a \mathbin{\underline{\circ}} a = a \circ a^I = e$

(b) $a \mathbin{\underline{\circ}} e = a \circ e^I = a \circ e = a$

(c) $a \mathbin{\underline{\circ}} a^I = a \circ (a^I)^I = a \circ a$

(d)	$a \underline{o} c = a \underline{o} c$	Reflexivity of equality
	$= b \underline{o} c$	Replacement
(e)	$c \underline{o} a = c \underline{o} a$	Reflexivity of equality
	$= c \underline{o} b$	Replacement
(f)	$a \underline{o} c = b \underline{o} c$	Assumption
	$a \circ c^{I} = b \circ c^{I}$	Definition of $\underline{o}$
	$a = b$	Right cancellation
(g)	$c \underline{o} a = c \underline{o} b$	Assumption
	$c \circ a^{I} = c \circ b^{I}$	Definition of $\underline{o}$
	$a^{I} = b^{I}$	Left cancellation
	$a^{I} \circ a = b^{I} \circ a$	Right operation
	$e = b^{I} \circ a$	Definition of a^{I}
	$b \circ e = b \circ (b^{I} \circ a)$	Left operation
	$b = (b \circ b^{I}) \circ a$	Definition of e and associativity
	$b = e \circ a$	Definition of b^{I}
	$b = a$	Definition of e
	$a = b$	Equality is symmetric
(h)	$(a \underline{o} b) \circ b = (a \circ b^{I}) \circ b$	Definition of $\underline{o}$
	$= a \circ (b^{I} \circ b)$	Associativity
	$= a \circ e$	Definition of b^{I}
	$= a$	Definition of e
(i)	$(a \circ b) \underline{o} b = (a \circ b) \circ b^{I}$	Definition of $\underline{o}$
	$= a \circ (b \circ b^{I})$	Associativity
	$= a \circ e$	Definition of b^{I}
	$= a$	Definition of e
(j)	$a \circ (b \underline{o} c) = a \circ (b \circ c^{I})$	Definition of $\underline{o}$

$= (a \circ b) \circ c^I$ Associativity

$= (a \circ b) \underline{\circ}\, c$ Definition of $\underline{\circ}$

(k) $a \,\underline{\circ}\, (b \circ c) = a \circ (b \circ c)^I$ Definition of $\underline{\circ}$

$= a \circ (c^I \circ b^I)$ Theorem 8

$= (a \circ c^I) \circ b^I$ Associativity

$= (a \,\underline{\circ}\, c) \,\underline{\circ}\, b$ Definition of $\underline{\circ}$

(l) $(a \,\underline{\circ}\, c) \,\underline{\circ}\, (b \,\underline{\circ}\, c) = (a \circ c^I) \circ (b \circ c^I)^I$

Definition of $\underline{\circ}$

$= (a \circ c^I) \circ ((c^I)^I \circ b^I)$

Theorem 8

$= (a \circ c^I) \circ (c \circ b^I)$

Theorem 7

$= (a \circ (c^I \circ c)) \circ b^I$

Associativity

$= (a \circ e) \circ b^I$

Definition of inverses

$= a \circ b^I$ Definition of e

$= a \,\underline{\circ}\, b$ Definition of $\underline{\circ}$

(m) $(a \,\underline{\circ}\, b) \circ (c \,\underline{\circ}\, d) = (a \circ b^I) \circ (c \circ d^I)$

Definition of $\underline{\circ}$

$= \{(a \circ b^I) \circ c\} \circ d^I$

Associativity

$= \{a \circ (b^I \circ c)\} \circ d^I$

Associativity

$= \{a \circ (c \circ b^I)\} \circ d^I$

(S,o) is a commutative group.

$= \{(a \circ c) \circ b^I\} \circ d^I$

Associativity

$= (a \circ c) \circ (b^I \circ d^I)$

Associativity

$= (a \circ c) \circ (d \circ b)^I$

Theorem 8

$= (a \circ c) \,\underline{\circ}\, (d \circ b)$

Definition of $\underline{\circ}$

$= (a \circ c) \,\underline{\circ}\, (b \circ d)$

The group is commutative.

(n) $(a \,\underline{\circ}\, b) \,\underline{\circ}\, (c \,\underline{\circ}\, d) = (a \circ b^I) \circ (c \circ d^I)^I$

Definition of $\underline{\circ}$

$= (a \circ b^I) \circ ((d^I)^I \circ c^I)$

Theorem 8

$= (a \circ b^I) \circ (d \circ c^I)$

Theorem 7

$= (a \circ d) \circ (c^I \circ b^I)$

Associativity and

Commutativity

$= (a \circ d) \circ (b \circ c)^I$

Theorem 8

$= (a \circ d) \,\underline{\circ}\, (b \circ c)$

Definition of $\underline{\circ}$

8. note $a = 0$ should be $a = 3$.

(a) a. $3 + (-3) = 0$

b. $3 + (-0) = 3$

c. $3 + -(-3) = 3 + 3 = 6$

d - g cannot be done since the assumptions are false.

h. $(3 + (-4)) + 4 = 3$

i. $(3 + 4) + (-4) = 3$

j. $3 + (4 + (-5)) = (3 + 4) + (-5)$

k. $3 + (-(4 + 5)) = (3 + (-4)) + (-5)$

l. $(3 + (-5)) + (5 + (-6)) = 3 + (-6)$

m. $(3 + (-4)) + (5 + (-6)) = (3 + 5) + (-(4 + 6))$

n. $(3 + (-4)) + (-(c + (-d))) = (3 + 6) + (-(4 + 5))$

(b) a. $3 \times \frac{1}{3} = 1$

b. $3 \times \frac{1}{1} = 3$

c. $3 \times \frac{1}{\frac{1}{3}} = 3 \times 3$

h. $(3 \times \frac{1}{4}) \times 4 = 3$

i. $(3 \times 4) \times \frac{1}{4} = 3$

j. $3 \times (4 \times \frac{1}{5}) = (3 \times 4) \times \frac{1}{5}$

k. $3 \times \frac{1}{4 \times 5} = (3 \times \frac{1}{4}) \times \frac{1}{5}$

l. $(3 \times \frac{1}{5}) \times \frac{1}{4 \times \frac{1}{5}} = 3 \times \frac{1}{4}$

m. $(3 \times \frac{1}{4}) \times (5 \times \frac{1}{6}) = (3 \times 5) \times \frac{1}{4 \times 6}$

n. $(3 \times \frac{1}{4}) \times \frac{1}{5 \times \frac{1}{6}} = (3 \times 6) \times \frac{1}{4 \times 5}$

2.11 Isomorphism (Time: 2 days)

The importance of this section is that it demonstrates to the students groups that have the same structure.

Isomorphism provides a means of studying the properties and proving propositions about unknown groups by examining the properties of known groups which are isomorphic to the unknown group.

2.12 Exercises

Exercises 1, 3 (d), and 4, should be demonstrated in class and the remaining exercises may be handled according to the teacher's discretion.

1. To show that $n \xrightarrow{f} 3^n$ is 1-1 from the group $(Z,+)$ into the group $(Q \setminus \{0\},o)$ we must show that f is defined on Z, which it is, and that if $f(x) = f(y)$ then $x = y$. In other word, that if $3^x = 3^y$ then $x = y$. An argument for this may be, if we assume that $3^z = 1$ if and only if $z = 0$,

$3^x = 3^y$	Assumption
$3^{-y} = 3^{-y}$	Reflexivity
$3^x \cdot 3^{-y} = 3^y \cdot 3^{-y}$	Right multiplication
$3^{x-y} = 3^0$	For all real numbers, a and b, $3^a \cdot 3^b = 3^{a+b}$
$3^{x-y} = 1$	Replacement $3^0 = 1$
$x - y = 0$	Assumption above
$x = y$	Right addition

$(Z,+)$ and $(Q\setminus\{0\},\cdot)$ are not isomorphic groups as no element of Z maps onto 2 of Q. For isomorphic groups the mapping must be onto. f is not onto.

Students may use other elements of Q to show that the mapping is not onto.

2. (a) The mapping f from Z_3 to {e,x,y}

$0 \longrightarrow e$

$1 \longrightarrow x$

$2 \longrightarrow y$

is 1-1 and onto. If the operation + is also replaced by o, the two tables become identical.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

o	e	x	y
e	e	x	y
x	x	y	e
y	y	e	x

(b) Use the mapping shown to display the isomorphism that converts one table into the other.

o	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

Δ	{ }	{a}	{b}	{a,b}
{ }	{ }	{a}	{b}	{a,b}
{a}	{a}	{ }	{a,b}	{b}
{b}	{b}	{a,b}	{ }	{a}
{a,b}	{a,b}	{b}	{a}	{ }

$o \longrightarrow \Delta$ $\quad e \longrightarrow \{\ \}$ $\quad x \longrightarrow \{a\}$ $\quad y \longrightarrow \{b\}$

$z \longrightarrow \{a,b\}$

(c) Let f be defined by $n \xrightarrow{f} 3^n$ for n in Z. Every n in Z has an image namely 3^n. Every 3^a has a pre-image, namely a. In exercise 1, we showed that f is 1-1. Hence, f is 1-1 and onto from Z to

$\{3^n: n \text{ in } Z\}$. We must show that f preserves products: that is, for every x in Z and y in Z

$$f(x + y) = f(x) \cdot f(y),$$

or equivalently that

$$3^{x+y} = 3^x \cdot 3^y$$

which is the case. (This may be proved by mathematical induction. At this level we assume this law for exponents.) It now follows, as $(Z,+)$ is a group, that $(\{3^n: n \text{ in } Z\},\cdot)$ is also a group, for every property possessed by $(Z,+)$ is imaged by an analogous property in $(\{3^n: n \text{ in } Z\}, \cdot)$.

It is an operational system: $3^x \cdot 3^y = 3^{x+y}$ for all x, y in Z.

$$(3^x \cdot 3^y) \cdot 3^z = 3^{x+y} \cdot 3^z = 3^{(x+y)+z} = 3^{x+(y+z)}$$
$$= 3^x \cdot 3^{(y+z)} = 3^x \cdot (3^y \cdot 3^z)$$

so that we have associativity.

$$3^0 \cdot 3^x = 3^{0+x} = \underline{\underline{3}}^x = 3^{x+0} = 3^x \cdot 3^0$$

for each x in Z, so that 3^0 serves as the identity element. The inverse of 3^x is 3^{-x} because

$$3^x \cdot 3^{-x} = 3^0 \quad\text{and}\quad 3^{-x} \cdot 3^x = 3^0.$$

3. (a) As Z_4 has 4 elements and Z_5 has 5, there can be no 1-1 mapping from Z_4 onto Z_5. There must be a 1-1 onto mapping between the sets of isomorphic groups.

(b) $(Z_6,+)$ can be generated by the single element 1:

$1 = 1$, $1 + 1 = 2$, $1 + 2 = 3$, $1 + 3 = 4$,
$1 + 4 = 5$, $1 + 5 = 0$.

$\{e,p,q,r,s,t\}$ has no element corresponding to 1. Hence, there is no mapping (1-1, onto) that preserves products.

(c) Each element of one group (the Klein 4 group) is its own inverse. This is not the case for the other group. Hence, there is no 1-1 onto mapping that preserves products.

(d) Suppose $(Q,+)$ and $(Z,+)$ were isomorphic groups. Let f be an isomorphism between them with Q the domain and Z the range. Let the pre-image of 1 in Z be $\underline{a}$ so that $f(a) = 1$. Then

$$\begin{aligned} 1 &= f(a) \\ &= f(\tfrac{a}{2} + \tfrac{a}{2}) \\ &= f(\tfrac{a}{2}) + f(\tfrac{a}{2}) \\ &= 2 \cdot f(\tfrac{a}{2}) \end{aligned}$$

Note: If a is in Q then $\frac{a}{2}$ is in Q.

Therefore, $f(\frac{a}{2}) = \frac{1}{2}$.

But f maps Q onto Z so that all images must be integers. $\frac{a}{2}$ is in Q and has $\frac{1}{2}$, which is not an integer, for its image under f. This contradiction shows that there is no isomorphism between $(Q,+)$ and $(Z,+)$ so that they cannot be isomorphic groups.

4. (a) Order of $(Z_4,+)$ is 4.

Order of 0 is 1.

Order of 1 is 4.

Order of 2 is 2.

Order of 3 is 4.

(b) Order of $(Z_5,+)$ is 5. Order of each element (except 0) is also 5. Order of 0 is 1.

(c) Order of $(Z_6,+)$ is 6. Order of 0 is 1. Order of 1 is 6, Order of 2 is 3, of 3 is 2, of 4 is 3, of 5 is 6.

(d) Order of $(Z_7,+)$ and each element (except 0) is 7.

2.14 Exercises (Time: 1 - 2 days)

Exercise 8 may be assigned for homework and a lesson built around this exercise can be developed the following day. Since there are many approaches and valid methods of proving exercise 8, the students should be encouraged to provide different proofs.

Before beginning exercise 10, the teacher should review the definition of an equivalence relation, presented in Course I.

If time permits, exercise 11 may be discussed in class. However, because of its degree of difficulty, it should not be assigned for homework.

1. (a) and (b) are groups. (c) and (d) are not groups.
2. 5! or 120.
3. (a) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix}^{I} = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 4 & 2\end{pmatrix}$

   (b) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 4 & 1 & 3\end{pmatrix} \circ \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 2 & 4\end{pmatrix}$
4. (a) 9 (b) 6 (c) 2
5. $(\{0\},+)$, $(\{0,4\},+)$, $(\{0,2,4,6\},+)$, $(\{0,1,2,3,4,5,6,7,8\},+)$. Yes.

6. **Solution Set**

(a) $\{s\}$

(b) $\{r\}$

(c) $\{r\}$

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

7. Let the **isomorphism** f from $(Z_6,+)$ to $(Z_7 \setminus \{0\},\cdot)$ be

$0 \longrightarrow 1$ $(=3^0)$

$1 \longrightarrow 3$ $(=3^1)$

$2 \longrightarrow 2$ $(=3^2)$

$3 \longrightarrow 6$ $(=3^6)$

$4 \longrightarrow 4$ $(=3^4)$

$5 \longrightarrow 5$ $(=3^5)$

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	1	3	2	6	4	5
1	1	3	2	6	4	5
3	3	2	6	4	5	1
2	2	6	4	5	1	3
6	6	4	5	1	3	2
4	4	5	1	3	2	6
5	5	1	3	2	6	4

If the symbols +, 0, 1, 2, 3, 4, 5 are replaced, maintaining order, by ·, 1, 3, 2, 6, 4 the first table becomes identical with the second. Hence, $(Z_6,+)$ and $(Z_7 \setminus \{0\}, \cdot)$ are isomorphic.

8. $(a \circ x) \circ b = c$ Assumption

$\Longleftrightarrow \{(a \circ x) \circ b\} \circ b^I = c \circ b^I$ Right multiplication

$\Longleftrightarrow (a \circ x) \circ (b \circ b^I) = c \circ b^I$ Associativity

$\Longleftrightarrow (a \circ x) \circ e = c \circ b^I$ Definition of b^I

$\Longleftrightarrow (a \circ x) = c \circ b^I$ Definition of e

$\Longleftrightarrow a^I \circ (a \circ x) = a^I \circ (c \circ b^I)$

Left multiplication

$\Longleftrightarrow (a^I \circ a) \circ x = a^I \circ (c \circ b^I)$

Associativity

$\Longleftrightarrow e \circ x = a^I \circ (c \circ b^I)$

Definition of a^I

$\Longleftrightarrow x = a^I \circ (c \circ b^I)$

Definition of e

Hence if there is a solution, it must be a $a^I \circ (c \circ b^I)$. Moreover, it is a solution as

$$a \circ \{a^I \circ (c \circ b^I)\} \circ b = (a \circ a^I) \circ c \circ (b^I \circ b)$$
$$= e \circ c \circ e$$
$$= c.$$

9. $(a \circ b^I)^I = (b^I)^I \circ a^I$ Theorem 8

$= b \circ a^I$ Theorem 7

10. It is not an equivalence relation. It is reflexive and transitive, but not symmetric.

11. $(y \circ a) \circ x = y \circ (a \circ x)$

$e \circ x = y \circ e$

$x = y.$

Thus $a \circ x = e$ and $x \circ a = e$; that is, every a has an inverse element, the only property remaining to complete the group structure.

Suggested Test on Groups

I. (P, *) is an operational system that **is** **associative**. Its table is given below. Answer all questions with regard to the table for (P, *).

*	a	b	c	d
a	c	a	d	b
b	a	b	c	d
c	d	c	b	a
d	b	d	a	c

(a) The identity element for this system is ______.

(b) a^I = __________.
b^I = __________.
c^I = __________.
d^I = __________.

(c) Is (P, *) a commutative system? ____________

(d) Is (P, *) a group? ____________

(e) If P is not a group, explain why. If P is a group, list one of the following choices as a subgroup of P: ({a,b}, *), ({b,c}, *), ({a,d}, *).

(f) Answer each by writing a single element:
d^2 = _______ a^3 = ________ b^8 = ________

II. Compute to show one permutation for each problem:

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ = ______________

(b) Find the inverse of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ = ________________.

(c) If $X = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$, find X^2. X^2 = ____________

(d) How many permutations can be made with the elements of the set {1,2,3,4,5}? __________

III. Consider the mappings: $f_1: n \longrightarrow n + 3$

$f_2: n \longrightarrow -n$

Compute the following within the set, Z:

(a) $f_1(0)$ = __________ (c) $f_1(f_2(5))$ = _________

(b) $f_2(-4)$ = __________ (d) $f_2(f_1(5))$ = __________

IV. Consider the groups of $(Z_3, +)$ and (R, o).

R = {a,b,c} and the operational table is:

o	a	b	c
a	b	c	a
b	c	a	b
c	a	b	c

(a) Are these two groups isomorphic? __________

(b) If your answer to "a" is yes, show why they are (exhibit the mapping and compare the operational tables). If your answer to "b" is no, show why not.

V. Fill in the proper reasons for the following proof of Left Cancellation in a group (S, o).

Theorem: If c o a = c o b, then a = b.

Statement	Reason
1. $c \circ a = c \circ b$	1.
2. $c^I \circ (c \circ a) = c^I \circ (c \circ b)$	2.
3. $(c^I \circ c) \circ a = (c^I \circ c) \circ b$	3.
4. $e \circ a = e \circ b$	4.
5. $a = b$	5.

VI. Prove:

(a) $(a \circ b^I)^I = b \circ a^I$ for a,b in a group

Statement	Reason
1.	1.

(b) Prove Right Operation in any group (S,o). Theorem: If $a = b$, then $a \circ c = b \circ c$.

Answers for Suggested Test of Groups

I. (a) b

(b) $a^I = d$

$b^I = b$

$c^I = c$

$d^I = a$

(c) yes

(d) yes

(e) $(\{b,c\}, *)$

(f) $d^2 = c$; $a^3 = d$; $b^8 = b$

II. (a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

(c) $X^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

(d) 5! or 120

III. (a) 3

(b) 4

(c) -2

(d) -8

IV. (a) yes

(b) There are 2 possible mappings.

1) $o \longrightarrow +$

$c \longrightarrow 0$

$a \longrightarrow 1$

$b \longrightarrow 2$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

o	c	a	b
c	c	a	b
a	a	b	c
b	b	c	a

2)

$o \longrightarrow +$

$c \longrightarrow 0$

$b \longrightarrow 1$

$a \longrightarrow 2$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

o	c	b	a
c	c	b	a
b	b	a	c
a	a	c	b

V.

	Statement	Reason
1.	$c \circ a = c \circ b$	1. Given (or Hypothesis)
2.	$c^I \circ (c \circ a) = c^I \circ (c \circ b)$	2. Left operation theorem
3.	$(c^I \circ c) \circ a = (c^I \circ c) \circ b$	3. Associativity of (S,o)
4.	$e \circ a = e \circ b$	4. Definition of c^I or inverse property of (S,o)
5.	$a = b$	5. Definition of e or identity property of (S,o)

VI. (a)

	Statement	Reason
1.	$(a \circ b^I)^I = (b^I)^I \circ a^I$	1. Theorem 8
2.	$= b \circ a^I$	2. Theorem 7

Students may elect to prove theorem 8 in this problem. But the use of the theorem is sufficient. Such a proof would require too much time in a one period test.

(b)

	Statement	Reason
1.	$a = b$	1. By assumption
2.	$a \circ c = a \circ c$	2. Equality is reflexive
3.	$a \circ c = b \circ c$	3. Substitution principle of equality (a = b).

Chapter 3

Affine Geometry

Time Estimate for Chapter: 18 - 21 days

The main objectives of the chapter are to:

1. introduce the student to an axiomatic affine geometry based on two incidence axioms and the Playfair version of the Euclidean parallel axiom.
2. prove theorems formally within this system and strengthen the notion of proof.
3. show that this axiomatic system can be given varied and sometimes unusual interpretations, whereby introducing the student to the notion of a model.
4. exhibit and analyze various finite and infinite models which despite their differences must nevertheless possess all the properties expressed by the axioms and theorems of our axiomatic geometry.
5. introduce the basic notion of parallel projection on which the idea of a coordinate system for the plane depends.
6. introduce the student to the concept of a vector.

General Remarks

Many of the exercises call for proofs of theorems. These should be regarded as a continuation of the content sections.

They are used in subsequent proofs and applications.

No special form or arrangement of proofs should be demanded of the students. Proofs presented in paragraph form should be completely acceptable if the reasoning is correct. The test of correctness is, of course, that each assertion made in a proof follow logically from the axioms and/or theorems previously deduced from the axioms.

Teachers should be careful to avoid discouraging the student when he submits a proof that is not correct. He should be praised for those parts of the proof that are sound. It takes time for a beginner to understand or appreciate the idea that he may use only properties of points, lines, etc. which he can deduce logically from the axioms and that he must avoid drawing conclusions which are based upon the appearance of a diagram.

If a student's proof or solution to a problem differs from yours (or ours) dc not assume that he is wrong. Have him explain because his answer may also be correct. Originality and creativity is a precious commodity to be nurtured and encouraged - not suppressed.

3.2 Axioms (And 3.1 Time: 1 1/2 - 2 days)

Stress that the plane π is a set of points and that lines are subsets of π. The set π and its subsets, the lines, are assumed to have certain properties which are expressed in the axioms.

The students should learn to state the axioms accurately.

In connection with the definition of parallel lines, be sure the students understand that every line is considered parallel to itself in this context. Here is a good opportunity to point out that definitions are man-made to serve some purpose. In the present case defining a line to be parallel to itself makes it possible to state axiom 3 and certain subsequent theorems simply, without awkward exceptions.

In connection with the term "affine geometry", stress that this is a geometry in which our Axioms 1, 2, and 3 hold.

3.3 Exercises

The purpose of these exercises is to clarify and sharpen the student's understanding of what the axioms assert as well as what they do not assert.

Exercise 3 affords an opportunity to point out that the axioms are "incomplete" in the sense that there are questions which cannot be decided one way or the other on the basis of just these three axioms. This point will become clearer when models are studied later. Thus, further axioms will have to be introduced later in order that the system of geometry shall have the properties we feel it should possess.

Exercise 4 illustrates that although a property may not be specifically assumed by any one of the axioms, it may be implied by several of them taken together. Thus the deduction of theorems from the axioms is anticipated by this exercise.

In Exercise 8, the students may recall that $m(A \cup B) = m(A) + m(B) - m(A \cap B)$. This sheds further light on the question raised there.

Question 10 anticipates the proof of one of the theorems in a later section. Do not require all students to do this problem at this time.

3.3 (Solutions to Exercises)

1. a) Yes. According t Definition 1, line M is considered parallel to line N even when M = N, i.e., when M "coincides" with N. In this case M and N have all their points in common.
   b) Yes.
2. Yes. Line m itself certainly contains E and is parallel to itself.
3. None. (Moreover, neither do they assert that a line may not contain three points. This question is left open by the axioms.)

4. No one of the axioms asserts this fact. (However it can be deduced from the 3 axioms. This is done later. See No.10 Sect. 3.5)

5. Axiom 1b. (A set that contains at least two points certainly contains at least one point.)

6. Axiom 2

7. None. (Note: Axiom 2 implies that there cannot be more than one such point, but none of the axioms guarantees that there must be such a point.)

8. Axiom 1 guarantees that each line contains at least two distinct points but it does not guarantee that the points in one line must be distinct from the points in another line. If a set A contains two members it does not follow that $A \cup B$ contains four members because A and B may share members in common.

9. Two distinct lines cannot intersect in more than one point because axiom 2 guarantees that only one line can contain two distinct points.

*10. By axiom 1a, π contains at least two lines; call them m and n. By axiom 1b m contains at least two points; call them A and B. Line n also contains two points (by axiom 1b). At least one of these two points must be different from either A or B, because axiom 2 stipulates that there is only one line containing A and B, namely m. Since n is not the same line as m, at least one of its points, call it C, must be different from either A or B. Hence there are at least three distinct points, namely, A, B, and C in plane π.

3.4 Some Logical Consequences of the Axioms (Time: 3 - 4 days)

This section introduces the first few theorems and definitions involving incidence properties of lines and planes. The proofs are presented in considerable detail and should be very carefully discussed in class. The role played by each axiom (or part of an axiom) should be clearly understood. Notice that axiom 3 (the Euclidean axiom) is not used in either of the proofs so Theorems 1 and 2 are strictly incidence theorems. Subsequent theorems which also require axiom 3 are affine theorems.

Be sure the student understands the meaning of the terms collinear, non-collinear, concurrent and non-concurrent.

One suggestion for this section is to ask students to rewrite the proofs of Theorems 1 and 2 in a "two-column" form, as an assignment.

3.5 Exercises

Skill in constructing proofs takes time to acquire. Hence the first two exercises (Theorems 3 and 4) supply most of the proof and the student is merely asked to supply reasons.

Let the student work on each of these exercises by himself either in class or via a homework assignment. Then have various students present their proofs for discussion, criticism, and correction of errors. A Special section in each student's notebook might be reserved for listing the theorems, each accompanied by a correct proof.

Allow various types of proofs, as noted under "General Remarks" at the beginning of the Commentary for this Chapter; encourage indirect, as well as direct proofs.

Allow variations in wording of the proofs so long as the reasoning is correct. Also, in proving a theorem stress use of previously proved theorems. This will make for still further variation in correct (and incorrect) proofs submitted by students. Scrutiny of such alternate proofs to check their validity is one of the best ways for a student to grow in mathematical maturity.

The students will probably find Exercise 7 quite difficult at this stage and Exercises 9 and 10 very hard. Nevertheless let them *try* them. All these theorems should be included in each student's list with proofs supplied by the better students, or by the teacher if necessary.

3.5 (Solutions to Exercises)

1. (1) axiom 1a
   (2) axiom 1b
   (3) axiom 2
2. (1) By axiom 1a
   (2) By axiom 1b

(3) If B and C were the same point then by axiom 2, m and n would be the same line (because m and n also contain point A).

(4) By Axiom 2.

(5) By an axiom, m is the only line containing both A and B and n is the only line containing both A and C. The 3rd line cannot contain A, since if it did, m and n would not be distinct lines.

3. (Theorem 5)

If A is a point in plane π, there are at least two lines in π, each containing point A.

Proof: By Theorem 4, there is a line, say m in π which does not contain A. By axiom 1b, m contains at least two points B and C. By axiom 2, there is a line r containing A and B and there is also a line s containing A and C, such that $r \neq s$. (Since if r were equal to s, B and C would be in r, in s, and in m, so that by Axiom 2, $r = s = m$. Then $A \in m$. But this contradicts the first statement.) Therefore r and s are the required lines.

4. (Theorem 6)

There are at least three non concurrent lines in the plane π.

Proof: By Theorem 2 there are at least three points in plane π, not all in the same line. Call these points A, B, C. By Theorem 5 there must be at least two lines m and n each containing A. By axiom 2 there is a line ℓ containing points B and C. This line ℓ cannot also contain A because A, B, and C are non-collinear. Hence this line ℓ must be distinct from lines m and n. By axiom 2, m and n cannot both contain a common point other than A, and since ℓ does contain A, ℓ, m, n are not concurrent.

5. (Theorem 7)

If each of two lines in π is parallel to the same line in π, then they are parallel to each other.

Proof: If $m \parallel s$ and $n \parallel s$ by hypothesis, we must prove $m \parallel n$. Suppose $m \nparallel n$. Then $m \neq n$, and $m \cap n$ contains exactly one point, say A. (Def. of $\parallel$ lines.) Hence $A \in m$ (where $m \parallel s$), and $A \in n$ (where $n \parallel s$). Since we already have $m \neq n$, we have a contradiction to axiom 3, which says that there is one and only one line containing a given point and $\parallel$ to a given line. Hence $m \parallel n$.

6. (Theorem 8)

If m is any line in π, then there are at least 2 points in π which are not in m.

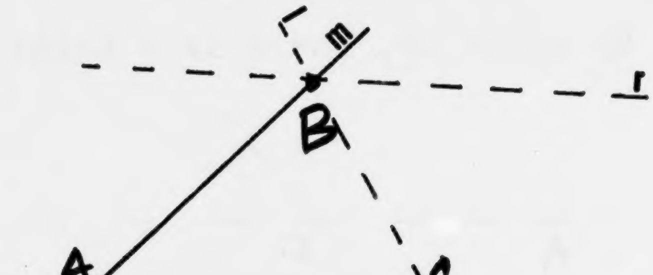
Proof: If m is in π, by Thoorem 1, there is a point in π, say A, not in m. By axiom 3, there is a line, r, containing A and $\parallel$ m. By axiom 1b, r contains at least 2 points, A and some other point C. This point $C \notin m$, since if $C \in m$, then $m \nparallel r$. But that would contradict the fact that $r \parallel m$. Hence A and C are the required points.

We include here two proofs for Exercise 7.

7. (Theorem 9)

If A is any point in plane π, then there are at least two lines in π which do not contain A.

1st Proof: If A is any point in plane π then by Theorem 5, there are at least two lines in π each containing point A. Call these lines m and n. By axiom 1b each of the lines m and n contains an additional point distinct from A. Call these points B and C, respectively. B and C must be distinct points because otherwise (if B = C) m and n would both contain the distinct points A and B, and by axiom 2, m and n would not be distinct lines. Since B and C are distinct points, there exists a line ℓ in π containing B and C (axiom 2).



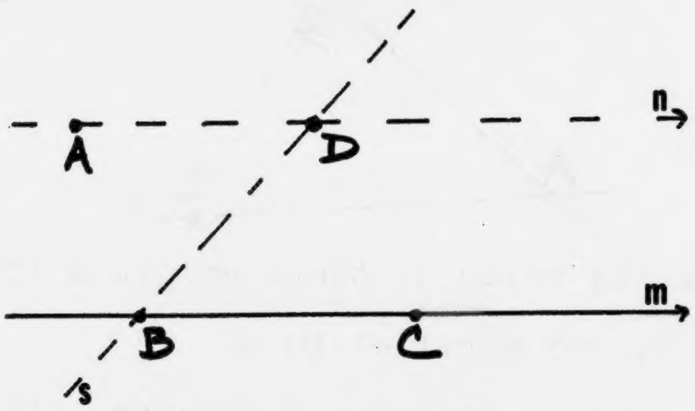
Moreover, by axiom 3, there exists a line r in π containing B, and parallel to n.

Neither r nor ℓ contains A because: (1) if r contained A then r = n (by axiom 3) and since r also contains B, then r = m (by axiom 2) and hence m = n contradicting $m \neq n$; (2) similarly, if ℓ contained A, then since ℓ contains B, $\ell = m$ (by axiom 2) and since ℓ contains C, $\ell = n$ (by axiom 2) from which once again $m = n$, a contradiction. Finally, to prove that r and ℓ are distinct lines we observe first, that $\ell \neq n$ (because n contains A while ℓ does not) and second, that ℓ and n both contain C. Therefore $\ell \nparallel n$, while on the other hand $r \parallel n$. Consequently $\ell \neq r$.

7. (Theorem 9)

If A is any point in plane π, then there are at least two lines in π which do not contain A.

2nd Proof: If A is any point in π then, by Theorem 4, there is a line in π which does not contain A; call this line line m. By axiom 1b, m contains at least two points, call them B and C. By axiom 3, there is a line in π which contains A and is parallel to m; call this new line n. By axiom 1b, there is a point in n other than A; call this new point D. By axiom 2, there is a line in π containing B and D; call this line s. We shall now prove that m and s are two (distinct) lines that do not contain A.



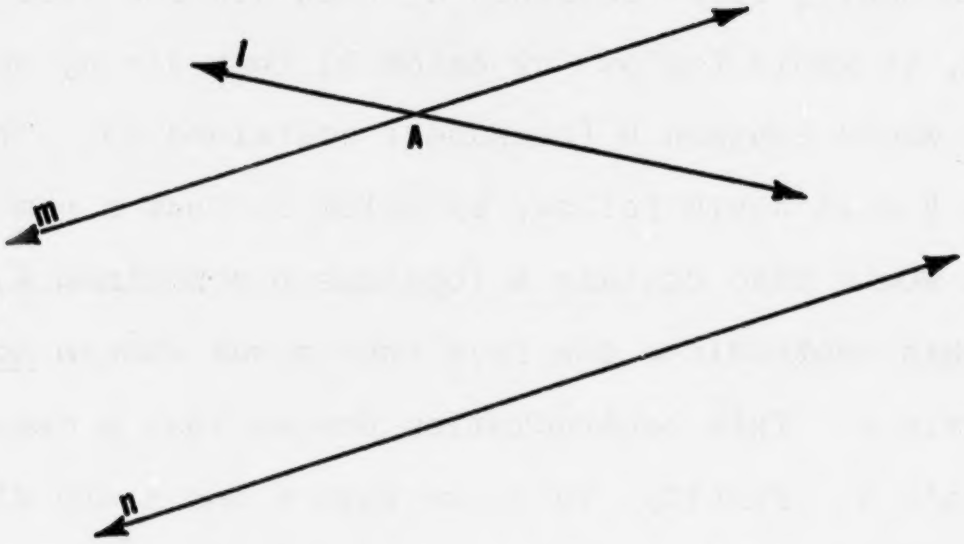
First of all m was chosen so as not to contain A. Secondly, if s contained A, then since s also contains D, it would follow (by axiom 2) that s = n, and hence n would contain B (because s contained B). Then since n $\|$ m it would follow, by axiom 3, that n = m and thus m would also contain A (because n contained A).
This contradicts the fact that m was chosen not to contain A. This contradiction proves that s cannot contain A. Finally, tp prove that m and s are distinct lines we observe that if m = s, then m would contain D (because s contains D) and since n $\|$ m, and n also contains D, it would follow from axiom 3 that n = m. But this would mean that m and n are not distinct lines, and that would contradict the fact that m was chosen to be distinct from n. Hence m and s are the required lines.

We include here two proofs for Exercise 8 - one is a direct proof and the other an indirect proof.

8. (Theorem 10)

If ℓ, m and n are lines in π such that m is parallel to to n, then if ℓ is not parallel to m, it follows that ℓ is not parallel to n.

1st Proof: Since ℓ is not parallel to m, it follows that ℓ and m are distinct lines and contain a common point, say A. Since m is parallel to n, it follows by axiom 3 that



any other line containing A cannot also be parallel to n. Hence ℓ is not parallel to n.

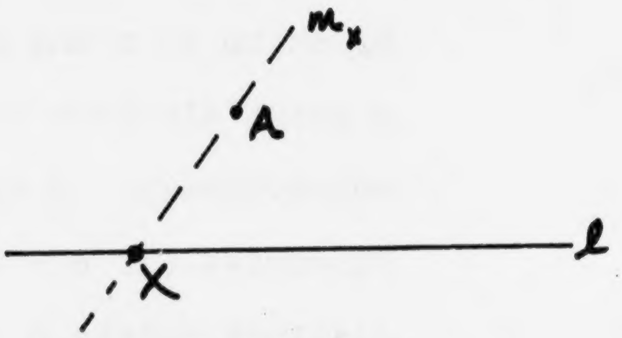
8. (Theorem 10)

   2nd Proof: ℓ, m, n are lines in π such that m $\parallel$ n. Assume $\ell \nparallel$ m and $\ell \nparallel$ n. We know $\ell \neq$ n, since $\ell \nparallel$ m and n $\parallel$ m. Further if $\ell \nparallel$ m, then $\ell \neq$ m and ℓ and m intersect in some point, call it A. But this contradicts axiom 3 since both ℓ and m would contain A and be $\parallel$ to n. Hence our assumption is false and $\ell \nparallel$ n.

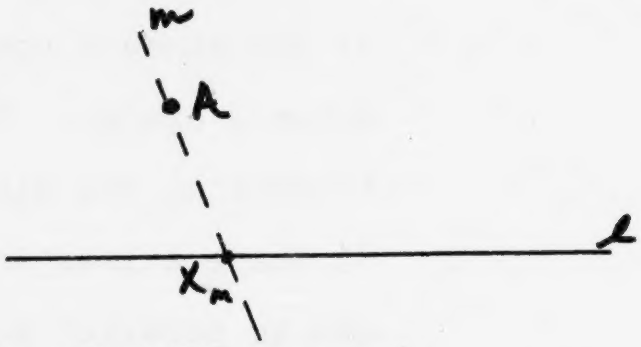
9. (Theorem 11)

   If ℓ is any line in plane π and A is any point in π which is not in line ℓ, then there is a one-to-one correspondence between the set of all points in ℓ and the set of all lines in π which contain A and are not parallel to ℓ.

Proof: In order to establish the one-to-one correspondence, we must "match" each point of ℓ with a unique line containing A but not parallel to ℓ, and conversely we must match each such line with a corresponding unique point of ℓ.



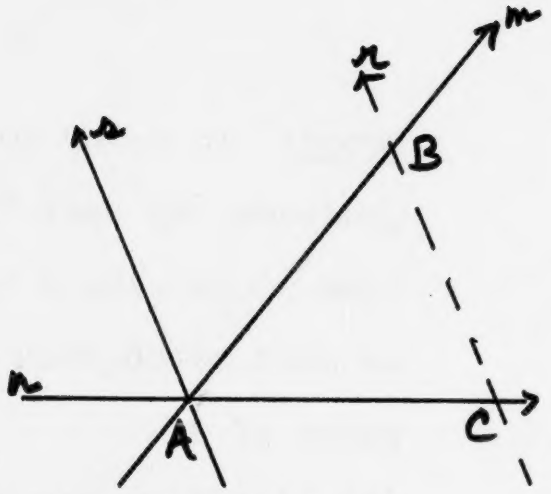
(1) Let X be any point $\in \ell$. Since A is not in ℓ, A and X are distinct points. Hence, by axiom 2 there is one and only one line in π, call it m_X, which contains both A and X. Thus, to each point X in ℓ there corresponds a unique line m_X which contains A and is not parallel to ℓ (because it contains a point of ℓ, namely X).



(2) Conversely, let m be any line containing A and not parallel to ℓ. Since m is not parallel to ℓ, m contains at least one point of ℓ. Moreover by Theorem 3 there is only one such point because m is distinct from ℓ (since m contains A while ℓ does not). Hence to each line m which contains A and is not parallel to ℓ there corresponds a unique point X_m in ℓ. This completes the proof.

10. (Theorem 12)

If A is any point in plane π, then there are at least three distinct lines in π each containing A.



Proof: If A is any point in π, then by Theorem 5 there are at least two (distinct) lines in π which contain A; call these distinct lines m and n. By axiom 1b m and n each contain a point distinct from A: call these points B and C, respectively. B and C are distinct points because otherwise (if B = C) m and n would both contain the distinct points A and B and then, by axiom 2, m and n would not be distinct lines. Since B and C are distinct points, there is a line r in π, which contains both B and C. r does not contain A because if it did, then since r contains B we would have r = m (by axiom 2) and since r contains C we would have r = n...(by axiom 2 again.. Therefore we would have m = n contradicting the distinctness of m and n. Now by axiom 3, there exists in π a line s which contains point A and is parallel to r. This line s is distinct from both m and n because each of these lines is not parallel to r (each contains a point in r as well as a point A not in R) while s is parallel to r. Hence s, m and n are three distinct lines in π, each containing A.

3.6 A Non-Geometric Model of the Axioms (Time: $1\frac{1}{2}$ - 2 days)

The example given here has been constructed with great care. It describes a "non-mathematical" situation which actually turns out to be startlingly mathematical! As the student studies the models in this section and in the next two sections, he should gain a deeper understanding of the power and value of abstraction in mathematics. Some of the models in this section and section 3.8 may be omitted with a good class.

3.7 Exercises

All of the exercises in this section should be covered. Exercises 1 - 8 give excellent practice in interpreting and in applying the theorems of our axiomatic geometry to the commando squad model. Exercise 9(a) (Theorem 13) should be added to the students notebook list, along with a correct proof. Exercise 9(b) throws further light on the two preceding exercises.

3.7 (Solutions to Exercises)

1. Theorem 1 translates into: For each team in the commando squad, there is at least one commando who does not belong to that team. [This can be re-phrased: No team includes all of the squad.]

   Theorem 2 translates into: There are at least three commandos in the commando squad who do not belong to the same team.

2. Theorem 4 translates into: For each commando in the commando squad there is a team to which he does not belong. [This is equivalent to: no commando belongs to all the teams.]
3. No. Theorem 6 asserts that there exist at least three lines in π which do not contain a common point. This translates into: There exist at least three teams in the commando squad which do not have a commando in common.
4. Theorem 8 translates into: For each team in the commando squad, there are at least two commandos in the squad who are not on that team.
5. Interpreting Theorem 10 we obtain the following: If ℓ, m and n are teams in the commando squad such that m has no commando in common with n, then if ℓ has a commando in common with m it follows that ℓ has a commando in common with n.
6. Theorem 11 translates as follows: If ℓ is any team in the commando squad and if A is any commando who is not a member of team ℓ, then there is a one-to-one correspondence between the set of all commandos in ℓ and the set of all teams which contain A and also contain a member of ℓ.

   Now these teams, each of which contains A along with one member of team ℓ, are distinct teams (because A is not a member of ℓ). Since there is one such team for each member of ℓ, the number of these teams must be the the same as the number of commandos in ℓ. However, there is one more team containing A, in addition to these teams, namely the "parallel" team (guaranteed by axiom 3) which contains A but does not contain any members of ℓ. Thus there is actually one more team containing A than there are commandos in ℓ.

Note: In this argument, A can be any commando in the squad, because for each such man, Theorem 4 guarantees that there is always at least one team to which he does not belong (see Exercise 2 above).

7. Requirements 1a, 1b and 2 are satisfied. Requirement 3 is not satisfied because no two of the teams are completely distinct. Every pair of two teams has a commando in common.

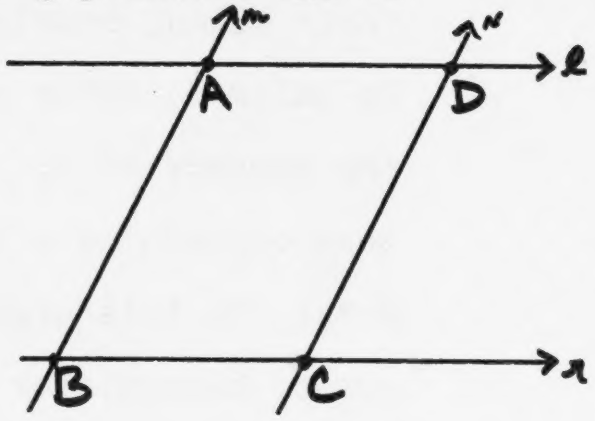
8. The following six teams will fill the bill:

Team 1: {Jones, Kelly}
Team 2: {Jones, Levy}
Team 3: {Jones, Mason}
Team 4: {Kelly, Levy}
Team 5: {Kelly, Mason}
Team 6: {Levy, Mason}

9. (a) (Theorem 13)

There are at least four points in plane π, no three of which are collinear.

Proof: By Theorem 2, there are at least three non-collinear points in plane π; call them A, B, C. By axiom 2 there is a line m in π, containing points A and B and a line r, in π, containing points B and C. $m \neq r$ because A, B, C are non-collinear. By axiom 3, there is a line ℓ, in π, containing A such that $\ell \parallel r$ and there is a line n in π, containing C such that $n \parallel m$. Now m and r are distinct lines containing point B. Hence certainly $m \nparallel r$ and therefore $m \neq \ell$ (because $\ell \parallel r$). But m and ℓ both contain point A, hence $\ell \nparallel m$. But $m \parallel n$, so it follows from Theorem 10, that $\ell \nparallel n$, i.e., ℓ and n both contain a (unique) point D in ℓ.



A, D, and C are non-collinear since A and D are on ℓ, $\ell \parallel r$, and C is on r. Since $m \neq \ell$ (proved above) it follows that D, A and B are non-collinear. Since $n \nparallel \ell$ but $\ell \parallel r$, it follows, by Theorem 10, that $n \nparallel r$. Consequently D, B and C are non-collinear. Thus the four points A, B, C, D have the property that no three are collinear.

(b) Theorem 13 proves that every model which satisfies the requirements of axioms 1, 2 and 3 must contain at least four "points".

Since the commando squad of Exercise 7 contains only three "points" (i.e., commandos) it cannot satisfy all requirements. Observe also that the existence of a fourth point depended on axiom 3 so evidently the third requirement fails to hold for this model.

In Exercise 8, the commando squad consists of exactly four men. The addition of a fourth "point" makes it possible to fulfill the requirements but only by organizing the teams ("lines") so that no three commandos (points) are on the same team (no three points are collinear).

3.8 Other Models of the Axioms - Finite and Infinite (Time: 3 - 4 days)

This section explores models in greater depth. It introduces four-point and nine-point geometries along with interesting interpretations of these finite geometries. It also presents an infinite model in which the student begins to get a preliminary glimpse of "analytic geometry". However the "plane" defined here is still a far cry from the real euclidean plane of ordinary analytic geometry.

3.9 Exercises

Exercises 1 - 10 provide further experiences in setting up models, interpreting the theorems and applying them to these models. Exercise 11(a) (Theorem 14) should be added to the student's notebook list of theorems and proofs. Problem 9c should be starred and can be done better when studying Chapter 6.

3.9 (Solutions to Exercises)

1. (a) Assignment of Players to Tennis Matches

Match	Players Assigned
No. 1	Al, Bill
No. 2	Al, Carl
Nol 3	Al, Don
No. 4	Bill, Carl
No. 5	Bill, Don
No. 6	Carl, Don

(b) Using his first initial to name each player we have the following model:

Pioneer Club: {A, B, D, C}

Doubles Teams: {A,B}, {A,C}, {A,D}, {B,C}, {B,D}, {C,D}

Tennis Players: A, B, C, D

This is a four-point geometry with plane P = Pioneer Club
line = doubles team
point = tennis player

(c) Axiom 1(a) translates into:

The Pioneer Club is a set of tennis players and it contains at least two doubles teams.

Axiom 1(b) translates into:

"Each doubles team in the Pioneer Club is a set of at least two tennis players."

Axiom 2 translates into:
"For every two tennis players in the Pioneer Club, there is one and only one doubles team in the Pioneer Club containing these two tennis players."

Before translating axiom 3, define "parallel" doubles teams to mean teams that are either identical or completely distinct (disjoint). Axiom 3 then translates into:
For every doubles team m and tennis player E in the Pioneer Club, there is one and only one doubles team in the Pioneer Club containing E and parallel to m.

A glance at the model in (a) or (b) shows that the axioms are satisfied.

(d) (1) parallel lines = doubles teams that have no tennis player in common or are identical.

(2) collinear points = a set of tennis players all on the same doubles team.

(3) non-collinear points = a set of tennis players not all on the same doubles team.

(4) concurrent lines = distinct doubles teams which have a tennis player in common.

Note: In this particular model there are three pairs of parallel doubles teams; any two tennis players are collinear, but three or more tennis players are non-collinear and there are four sets of concurrent lines.

(e) Theorem 1 translates into:
For each doubles team in the Pioneer Club there is a tennis player in the Pioneer Club who is not on that team.

Theorem 2 translates into:
There are at least three tennis players in the Pioneer Club who are not all on the same doubles team.

Theorem 3 translates into:
Two distinct doubles teams cannot have more than one tennis player in common.

Therrems 2 and 3 are trivial because a doubles team consists of exactly two tennis players.

(f) Theorem 5 asserts that each point is contained in at least two distinct lines (hence each player will participate in at least two matches). Theorem 9 asserts that for each point there are at least two lines which do not contain A (hence for each player there will be at least two matches in which he will not participate).

2. (a) Yes; yes; no

(b) No. Every two distinct lines have a point in common.

(c) non-collinear

(d) Theorem 1: valid

Theorem 3: valid Depend only on axioms 1 and 2

Theorem 5: valid

Theorem 8: not valid

Depend on axiom 3

Theorem 9: not valid

Theorem 1 valid (in the vacuous sense that there are no parallel lines)

3. (a) Plane π: {A,B}

Line : {A,B} (note: only one line)

Points : A,B

Axiom 1a is not satisfied.

Axiom 1b is satisfied.

Axiom 2 is satisfied.

Axiom 3 is obviously satisfied because plane π does not contain a line and a point not on that line and the only case is when the point is on the line.

(b) See proof of Theorem 13 (Exercise 9(a) in section 4.7)

4. (a) There are actually 10 lines.

(b) Each line contains exactly 2 points.

(c) (1) not parallel

(2) not parallel

(3) parallel

(4) parallel

(d) yes

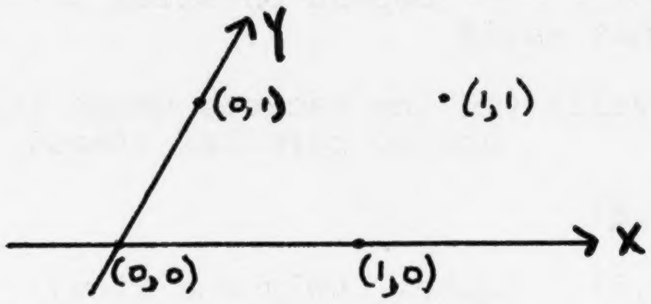
(e) no. For example, there are two lines containing point C, namely {A,C} and {B,C}, both of which are parallel to line $\{D,E\}$.

(f) Two.

5. Any four-point model (e.g. the tennis club model of exercise 1 above) shows this.

6. (a) Points in plane π: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$.

(b)



(c) Equations of lines in π are:

ℓ_1: $1x + 0y = 0$

ℓ_2: $0x + 1y = 0$

ℓ_3: $1x + 0y = 1$

ℓ_4: $0x + 1y = 1$

ℓ_5: $1x + 1y = 0$

ℓ_6: $1x + 1y = 1$

(d) Lines which correspond are:

ℓ_1: $\{(0,0), (0,1)\}$

ℓ_2: $\{(0,0), (1,0)\}$

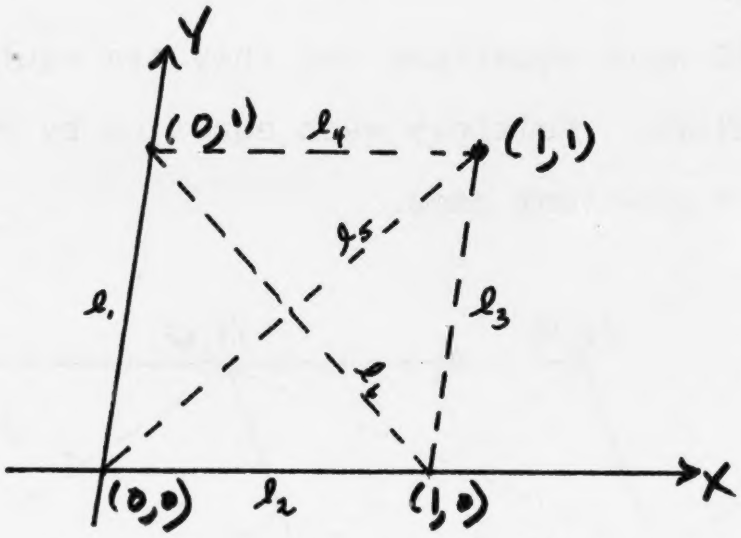
ℓ_3: $\{(1,0), (1,1)\}$

ℓ_4: $\{(0,1), (1,1)\}$

ℓ_5: $\{(0,0), (1,1)\}$ Note: $1 + 1 = 0$

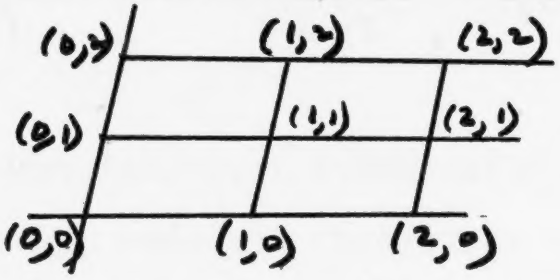
ℓ_6: $\{(1,0), (0,1)\}$

(e)


Y
(0,1)
ℓ_4
(1,1)
ℓ_5
ℓ_1
ℓ_3
ℓ_6
(0,0)
ℓ_2
(1,0)
X

7. (a) Points: (0,0), (0,1), (0,2)
(1,0), (1,1), (1,2)
(2,0), (2,1), (2,2)

(b)

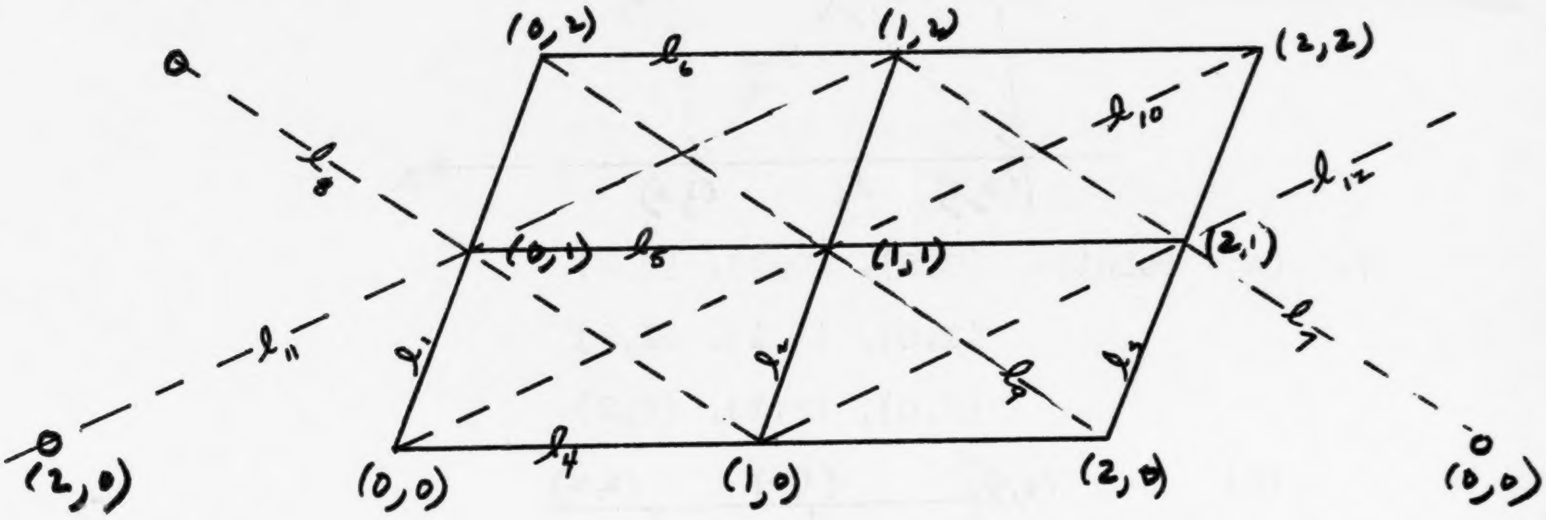

(0,2)
(1,2)
(2,2)
(0,1)
(1,1)
(2,1)
(0,0)
(1,0)
(2,0)

(c) ℓ_1: $1x + 0y = 0$
ℓ_2: $1x + 0y = 1$
ℓ_3: $1x + 0y = 2$
ℓ_4: $0x + 1y = 0$
ℓ_5: $0x + 1y = 1$
ℓ_6: $0x + 1y = 2$
ℓ_7: $1x + 1y = 0$
ℓ_8: $1x + 1y = 1$
ℓ_9: $1x + 1y = 2$
ℓ_{10}: $2x + 1y = 0$
ℓ_{11}: $2x + 1y = 1$
ℓ_{12}: $2x + 1y = 2$

(d) $\ell_1 = \{(0,0), (0,1), (0,2)\}$
$\ell_2 = \{(1,0), (1,1), (1,2)\}$
$\ell_3 = \{(2,0), (2,1), (2,2)\}$
$\ell_4 = \{(0,0), (1,0), (2,0)\}$
$\ell_5 = \{(0,1), (1,1), (2,1)\}$
$\ell_6 = \{(\ ,2), (1,2), (2,2)\}$
$\ell_7 = \{(1,2), (2,1), (0,0)\}$
$\ell_8 = \{(1,0), (0,1), (2,2)\}$
$\ell_9 = \{(1,1), (2,0), (0,2)\}$
$\ell_{10} = \{(0,0), (1,1), (2,2)\}$
$\ell_{11} = \{(0,1), (1,2), (2,0)\}$
$\ell_{12} = \{(0,2), (1,0), (2,1)\}$

There are 12 more equations but they are equivalent to the above equations. Multiply each equation by 2 and you will get these equivalent ones.

(e)



Note: The open circles don't represent new points. Each open circle represents the same point as the correspondingly labelled heavy dot. This diagram is a slight variation of Figure 14 in the text with a more symmetrical placement of the open circles.

8. (a) $1x + 0y = 0$: solution set: all ordered pairs of the form $(0,y)$

$0x + 1y = 0$: solution set: all ordered pairs of the form $(x,0)$

These two "lines" represent the y-axis and x-axis respectively. The "plane" $\pi = Z \times Z$ certainly contains both these "lines", i.e., both solution sets.

(b) There are many points that can be used to show this.

(1) $x - y = 0$: solution set contains the "points" $(0, 0)$ and $(1, 1)$

(2) $x + y = 2$: solution set contains the "points" (0, 2) and (2, 0)

(3) $2x - y = 0$: solution set contains the "points" (0, 0) and (1, 2)

(4) $3x + 4y = 5$: solution set contains the "points" (3, -1), and (-1, 2)

(c) (1) We first seek an equation of the form

$$ax + by = c$$

where a, b, c are integers and a, b are not both zero, and such that (0, 0) and (1, 1) are both in the solution set. Substituting (0, 0) for (x, y) we obtain $0 = c$. Substituting (1, 1) for (x, y) we obtain

$$a + b = c$$

Hence $a + b = 0$

$$\therefore\ a = -b$$

Hence any equation of the form $ax - ay = 0$ where $a \neq 0$ will fill the bill. Furthermore all equations of this form are equivalent to the simpler equation

$$x - y = 0$$

i.e., they all have the same solution set namely $\{(0,0), (1,1), (-1,-1), (2,2), (-2,-2)\ldots\}$ This solution set represents the unique "line" determined by the two points (0, 0), and (1, 1)

(2) For the points (4, -1) and (2, 0) we obtain

$$4a - b = c$$

$$\underline{2a \quad\quad = c}$$

$$\therefore \quad 4a - b = 2a$$

$$\therefore \quad b = 2a \text{ (where } a \neq 0)$$

Hence the equation must have the form

$$ax + 2ay = 2a \qquad (a \neq 0)$$

or equivalently

$$x + 2y = 2$$

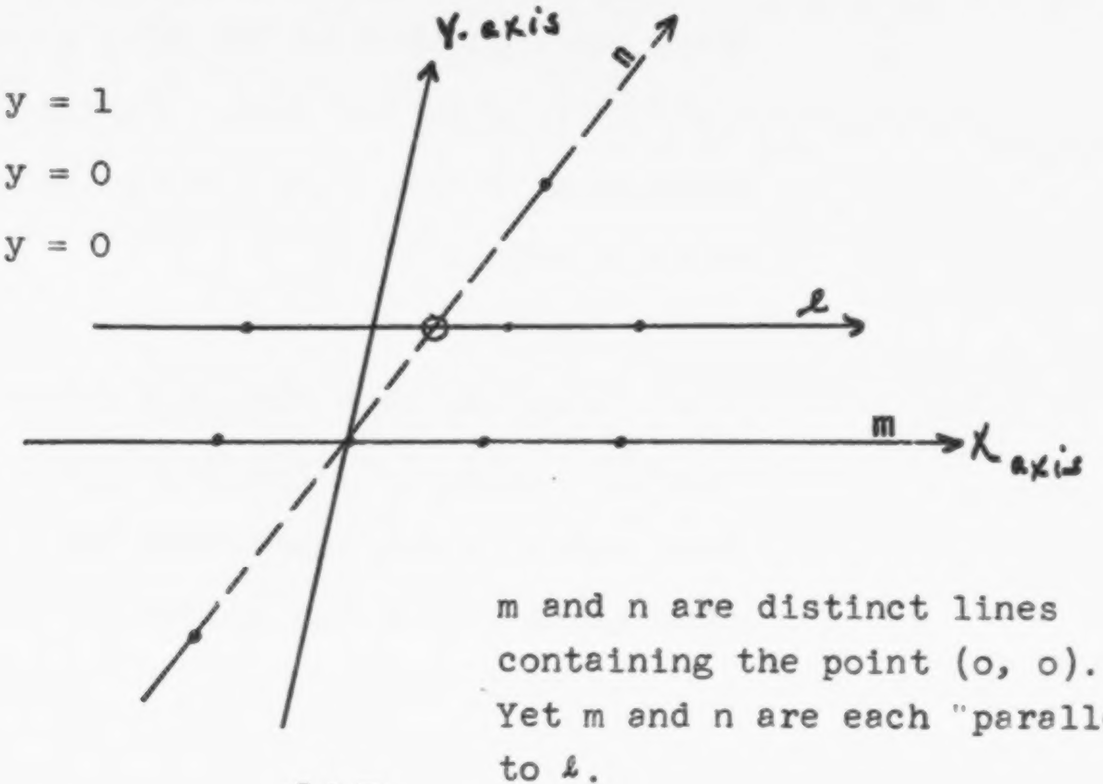
The solution set (in integers) for this equation represents the unique "line" determined by the two given points.

(d) Since the plane $\pi = Z \times Z$ is a set of lattice points we seek "lines" of lattice points for which the Euclidean parallel axiom (axiom 3) fails. Such "lines" are indicated in the diagram.

ℓ: $ox + y = 1$

m: $ox + y = 0$

n: $2x - y = 0$



m and n are distinct lines containing the point (o, o). Yet m and n are each "parallel" to ℓ.

9. (a) Yes. In fact the illustrations used in 8(b) are equally applicable in Q x Q because all ordered pairs of integers are contained in Q x Q as well.

(b) No. The counter example in 8(c) is no longer valid in Q x A because although line m is still parallel to line ℓ, line n is no longer parallel to ℓ. In fact lines n and ℓ intersect in the "point" (1/2, 1) as is readily seen by solving the two equations.

*(c) This problem should not be assigned until after Chapter 6, Section 6.15, problem 6. Then it can be done by the point-slope form of an equation. An alternate solution is given here.

Axiom 3 asserts that for every line m and every point E in the plane π, there is one and only one line containing E and parallel to m.

To verify this for the "line" m defined by

$$3x + 4y = 5$$

and the "point" E = (2, 1) we first observe that point E is not contained in line m because (2, 1) does not satisfy $3x + 4y = 5$.

Next we seek an equation

$$ax + by = c$$

where a, b, c are rational numbers, a and b not both 0, and such that this equation is satisfied by (2, 1) but is not satisfied by any of the solutions of $3x + 4y = 5$. Substituting (2, 1) in the equation $ax + by = c$ we obtain

$$ax + by = 2a + 1b$$

or $a(x - 2) + b(y - 1) = 0$. (This is clearly satisfied by (2, 1)). Now from the equation $3x + 4y = 5$ we obtain

$$x = \frac{5 - 4y}{3}$$

Substitute in the new equation:

$$a(\frac{5 - 4y}{3} - 2) + b(y - 1) = 0$$

$$a(\frac{-1 - 4y}{3} + b(y - 1) = 0$$

$$a(-1 - 4y) + 3b(y - 1) = 0$$

$$(3b - 4a)y = a + 3b$$

Now if $3b - 4a \neq 0$ this equation can be solved for y. Hence, since we want no solution for y, we must stipulate that

$$3b - 4a = 0$$

i.e. $b = \frac{4}{3}a$

where a may not be zero because otherwise both a and b would be zero. Substituting $b = \frac{4}{3}a$ in the equation $a(x - 2) + b(y - 1) = 0$ we obtain

$$a(x - 2) + \frac{4}{3}a(y - 1) = 0$$

and since $a \neq 0$, this equation is equivalent to

$$(x - 2) + \frac{4}{3}(y - 1) = 0$$

$$3(x - 2) + 4(y - 1) = 0$$

$$3x - 6 + 4y - 4 = 0$$

$$3x + 4y = 10$$

This equation defines the desired line. To check that it contains point E we substitute (2, 1) for (x, y):

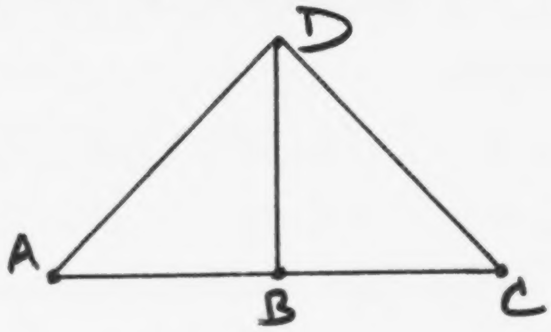
$$3(2) + 4(1) = 10$$
$$10 = 10$$

To check that it is parallel to line m we observe that the equations

$$3x + 4y = 5$$
$$3x + 4y = 10$$

are clearly inconsistent. No ordered pair (x, y) can satisfy both equations because $5 \neq 10$.

10. (a)



(b) By (2) there are at least two committees. One of these has exactly 3 members; call them A, B, C. The second committee has one member from each other committee. But by (1) this second committee must also contain another member D and this other member D must be different from B and C because of (3). Thus there are at least four people in the family. [Note that we did not use (4) in this proof but we did use it to set up the model. (4) requires that E and D form a committee and also C and D.]

(c) Each line is parallel to itself, but no two distinct lines are parallel because (3) stipulates that each line contains a point from each other line.

11. (a) (Theorem 14)

There are at least six lines in plane P.

Proof: By Theorem 13, there are four points in plane P no three of which are collinear. Call these points A, B, C, D. Since these points are distinct every two of these points determines a line by axiom 2. We can select two points out of the four in six ways:

$\{A, B\}$ $\{A, C\}$ $\{A, D\}$

$\{B, C\}$ $\{B, D\}$ $\{C, D\}$

The six lines thus determined must all be distinct, because no three of the points are collinear.

(b) The four point geometry exhibits exactly six lines.

3.10 Equivalence Classes of Parallel Lines (Time 1 - $1\frac{1}{2}$ days)

Before teaching this section it will be helpful to review with the students Section 8.15 of Course 1 (Equivalence Classes and Partitions). The particular equivalence classes introduced here are families of parallel lines. A brief descriptive term for each family is the phrase "parallel class of lines" or simply: "parallel class". Many mathematicians prefer to call these equivalence classes "directions".

Thus any pair of parallel lines are said to be "in the same direction", meaning "in the same equivalence class". In the text we have avoided using these other descriptive terms, but the teacher may choose to use one or more of them, if he feels it will be helpful.

Have the student interpret the equivalence classes in the various models he has studied. The exercises include a few such interpretations.

3.11 Exercises

Exercises 1 and 2 are useful for reviewing the three requirements for an equivalence relation (reflexivity, symmetry and transitivity). Exercises 3 and 4 provide practice in interpreting equivalence classes in specific models.

Exercise 5 (Theorem 16) should be added to the theorem list. Exercise 6 is a nice application of Theorem 10, although it can be proved without using Theorem 10.

3.11 (Solutions to Exercises)

1. (a) no (b) yes (c) no (d) yes (e) no
   (f) no (g) no (h) no (i) yes

   The equivalence classes are:

   (b) The subsets which contain all living people who are a given age

   (d) The sets of books having a specified number of pages

(i) The sets of students in a particular grade

2. (a) Reflexivity: Every book has the same author as itself.

   (b) Symmetry: If book x has the same author as book y, then book y has the same author as book x.

   (c) Transitivity: if x has the same author as y and y has the same author as z, then x has the same author as z.

   (d) The equivalence classes are the sets of all books by a particular author.

3. Equivalence classes for tennis club model of Exercise 1 Section 3.9:

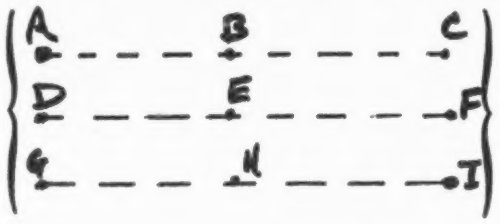
{(A, B), (C, D)}

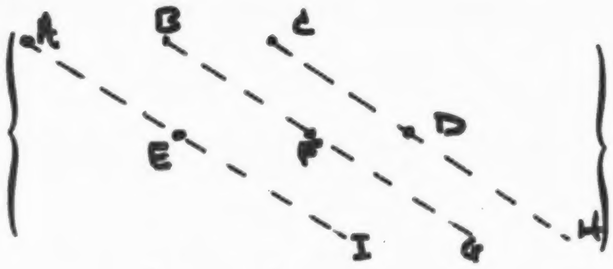
{(A, C), (B, D)}

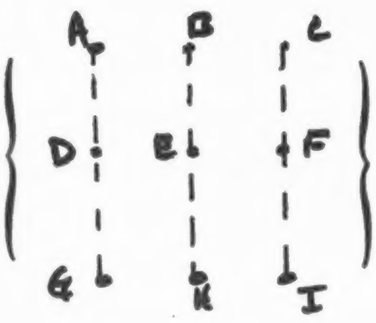
{(A, D), (B, C)}

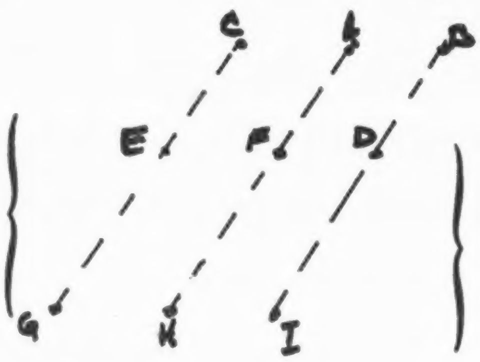
Note: These equivalence classes are the parallel lines for the relation, "is parallel to."

4. (a) For the nine point geometry,









(b)

$$\left\{\boxed{A,B,C}\quad\boxed{D,E,F}\quad\boxed{G,H,I}\right\}\qquad\left\{\boxed{A,E,I}\quad\boxed{B,F,G}\quad\boxed{C,D,H}\right\}$$

$$\left\{\boxed{A,D,G}\quad\boxed{B,E,H}\quad\boxed{C,F,I}\right\}\qquad\left\{\boxed{C,E,G}\quad\boxed{A,F\ H}\quad\boxed{B,D,I}\right\}$$

5. (Theorem 16)

There are at least three distinct equivalence classes in plane π.

Proof: By Theorem 12, if A is a point in plane π, then there are at least three distinct lines r, s, t in π, each containing A. Each of these lines determines the equivalence class of all lines in π parallel to that line. Call these equivalence classes E_r, E_s and E_t respectively. These equivalence classes are distinct, in fact disjoint, because if, say, E_r and E_t both contained a line ℓ, then r and t would be distinct lines each containing A and each parallel to ℓ in violation of axiom 3. Hence there are at least three distinct equivalence classes in plane π.

6. No; because if D contained n, then D would have to contain every line parallel to n and therefore would have to contain m, which it does not.

3.12 Parallel Projection (Time: 2 - 3 days)

The ideas introduced in this section are basic for much of the subsequent work in coordinate geometry. The section brings together a number of ideas the student has encountered in various other parts of the course. These include the notions of mapping, one-to-one correspondence, inverse mapping and equivalence class. These ideas, applied here, lead to the fundamental concept of a parallel projection from a line n onto a line m, on which much of geometry can be based.

Two interesting corollaries of Theorem 17 are: (1) all lines in P have the same number of points: and (2) if one line in P has infinitely many points, then every line π has infinitely many points. These corollaries are proved as follows:

Proof of (1): Let m and n be any two lines. By Theorem 17, there is a one-to-one correspondence between the points of m and the points of n. Hence m and n have the same number of points.

Proof of (2): Suppose a line m, in π, has infinitely many points. If n is any other line in π, there is a one-to-one correspondence between the points of m and the points of n (by Theorem 17). Therefore, since m has infinitely many points, n must also have infinitely many points. Care should be taken when speaking of a mapping D_m that the domain is stated. Sometimes the domain is the entire plane π, and sometimes it is restricted to a line n.

3.13 Exercises

These exercises provide practice with parallel projections and aim at discovery of some of the properties of parallel projections. The exercises are intended to be chiefly of an experimental and exploratory nature with the students formulating conjectures about parallel projections. No formal proofs should be called for in this set of exercises.

3.13 (Solution to Exercises)

1.

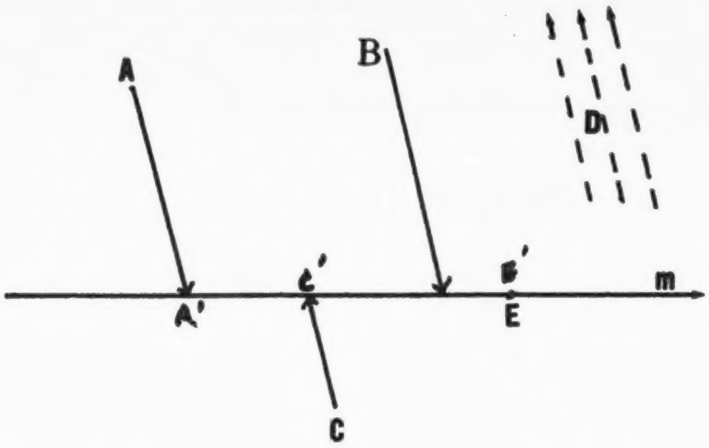
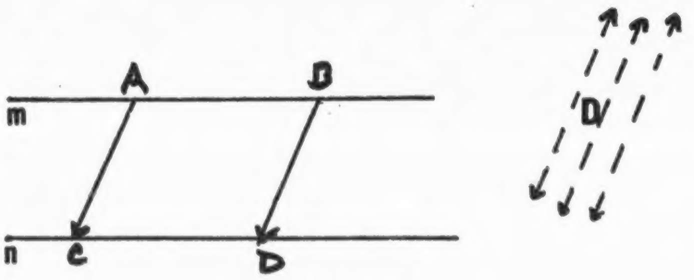


Image points indicated by A', B', C', D', E' (E = E')

2. (a) The point where n intersects m.

   (b) Each point of m maps on to itself.

3.

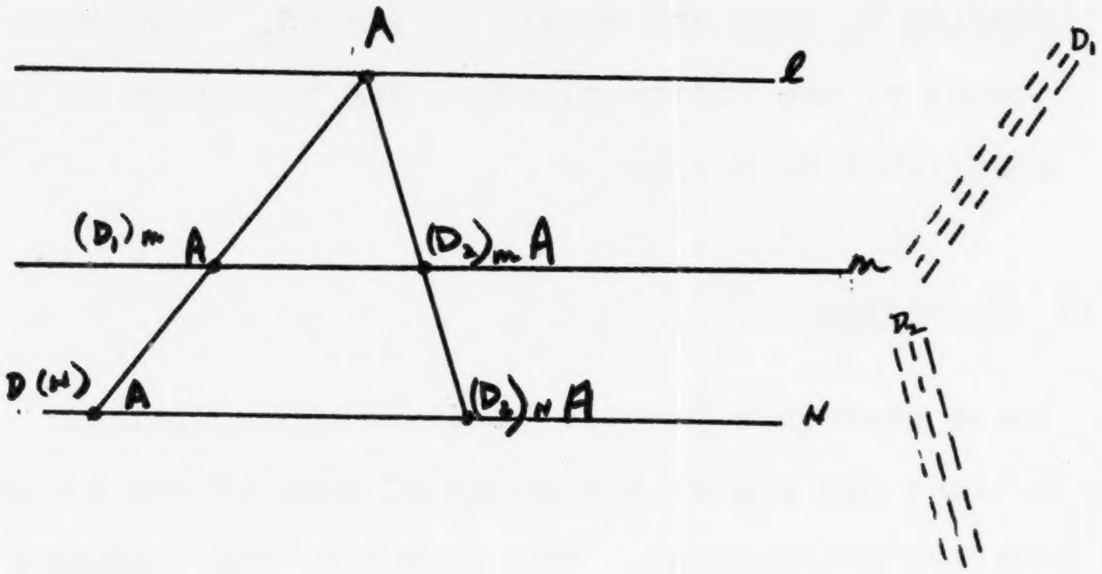


(a)

(b) A and B, respectively

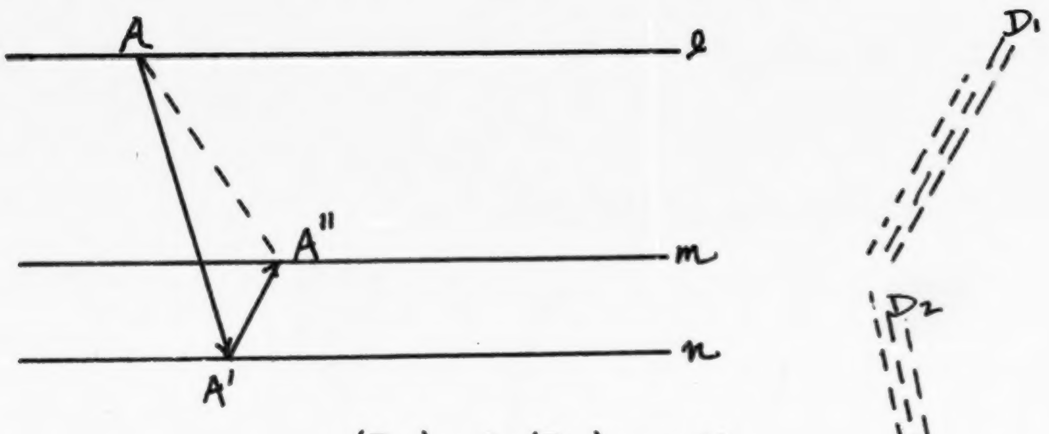
(c) They are inverse mappings.

4. (a)


A
l
(D₁)m A
(D₂)m A
m
D(n) A
(D₂)n A
n
D₁
D₂

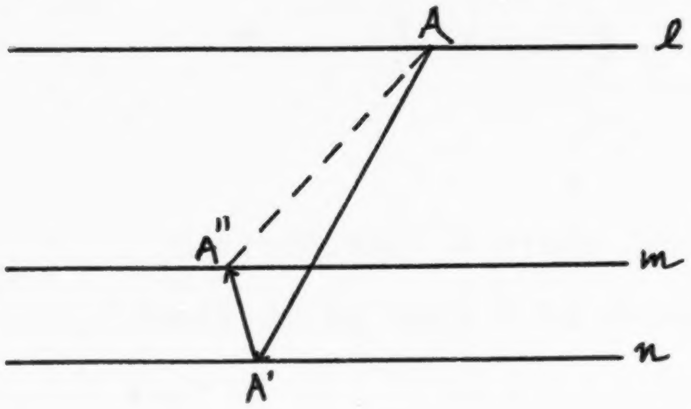
(b)

(1)


A
l
A''
m
A'
n
D₁
D₂

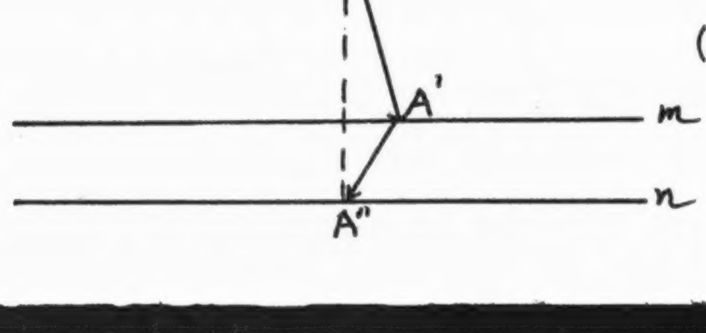
$(D_1)_m \circ (D_2)_n = A''$

(2)



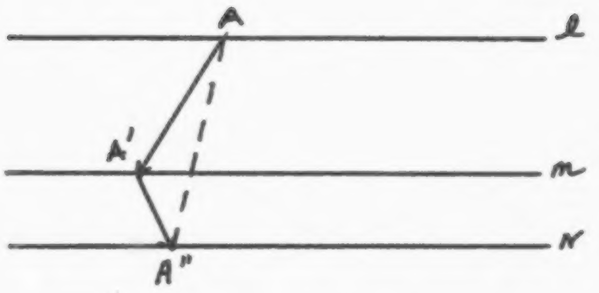
$(D_2)_m \circ (D_1)_n = A''$

(3)



$(D_1)_n \circ (D_2)_m = A''$

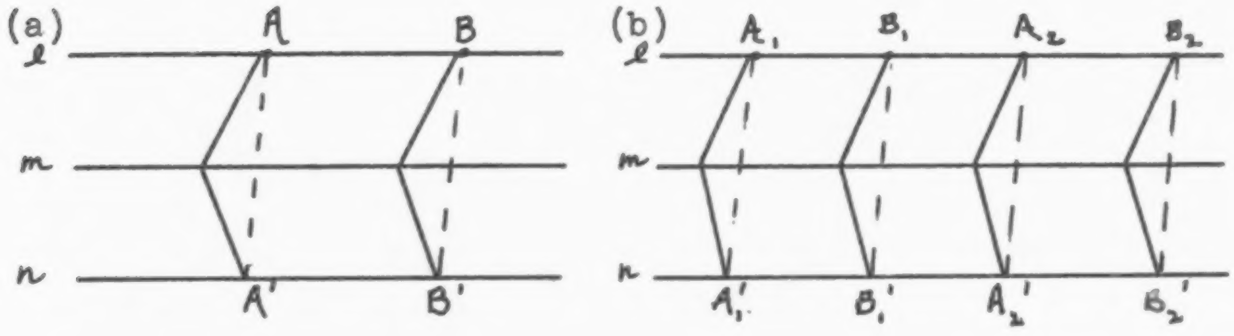
(4)



$(D_2)_n \circ (D_1)_m = A''$

4. (c) Similar diagram with another point B on ℓ. Composition of parallel projections appears to be non-commutative.

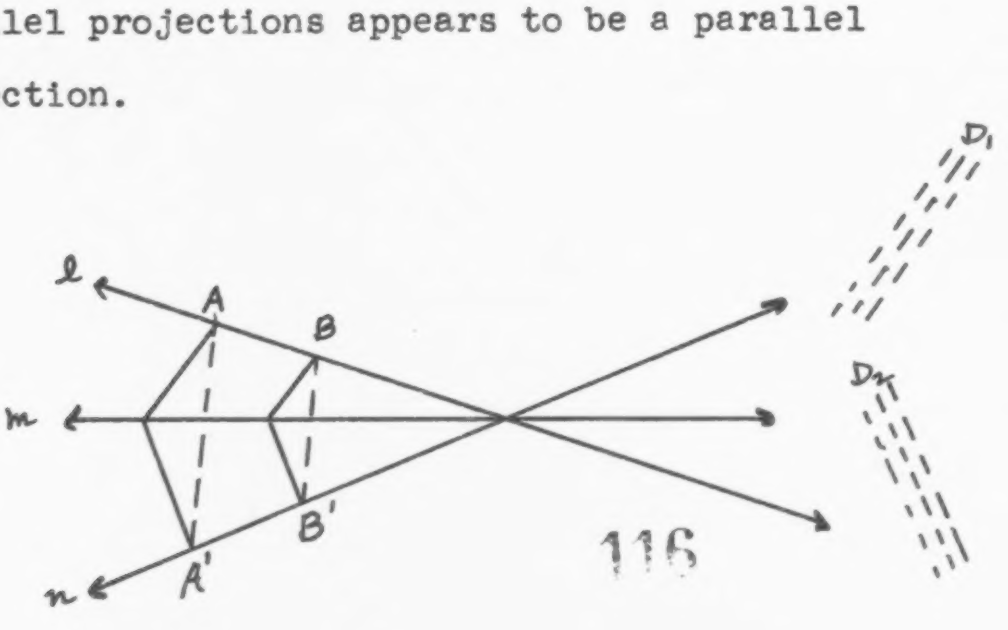
5.


(a)
A
B
l
m
n
A'
B'
(b)
A₁
B₁
A₂
B₂
l
m
n
A₁'
B₁'
A₂'
B₂'

Yes. In this case the composition of the parallel projections appears to be a parallel projection.

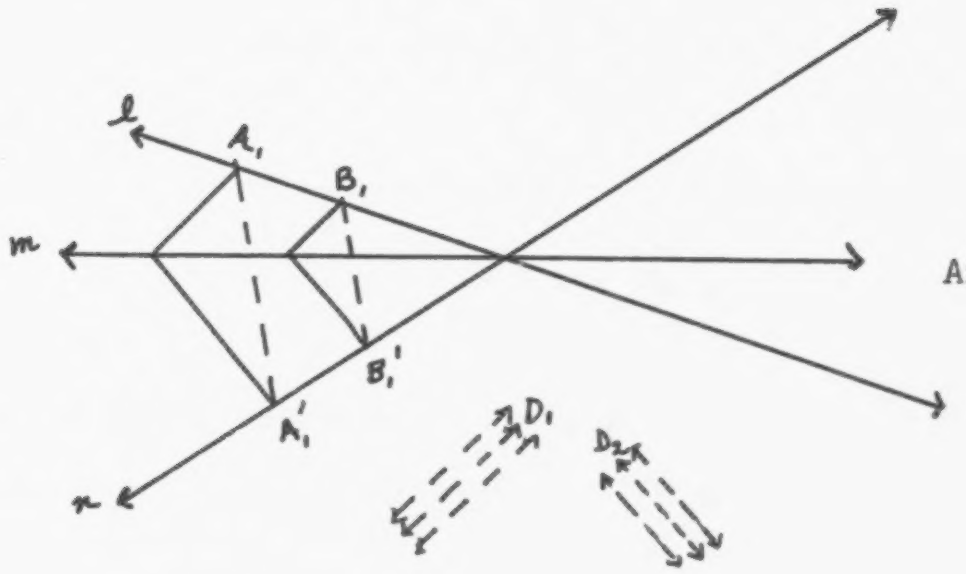
6.

(a)


l
A
B
m
B'
A'
n
D₁
D₂

6.

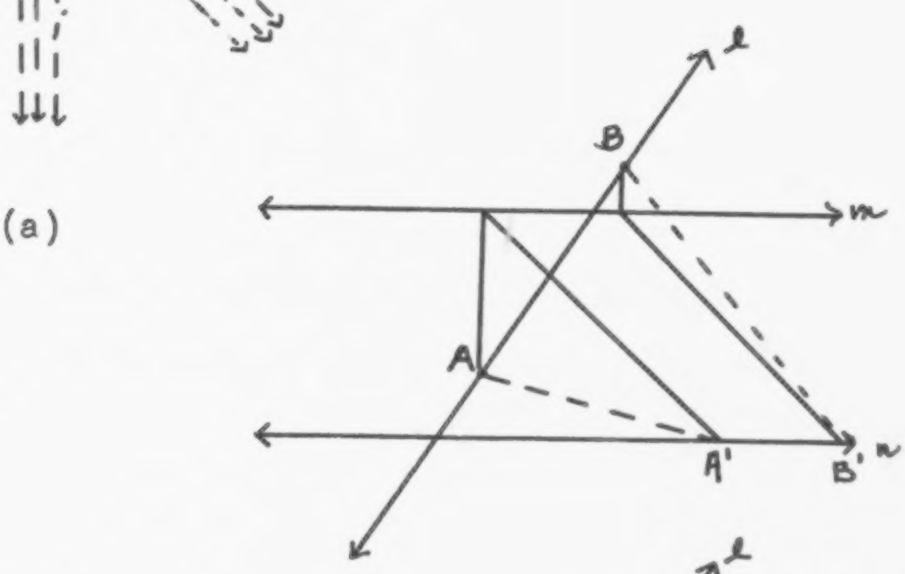
(b)


l
m
n
A₁
B₁
A₁'
B₁'
D₁
D₂

Ans. Yes

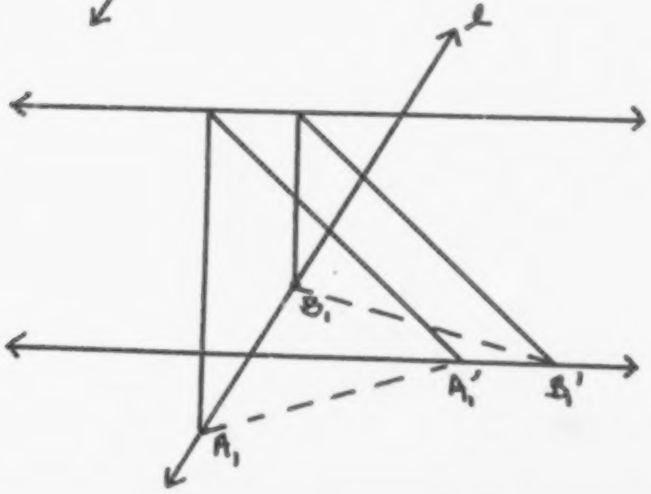
7.

(a)

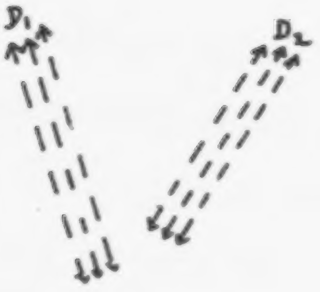

D₁
D₂
l
m
n
B
A
A'
B'

Ans. No

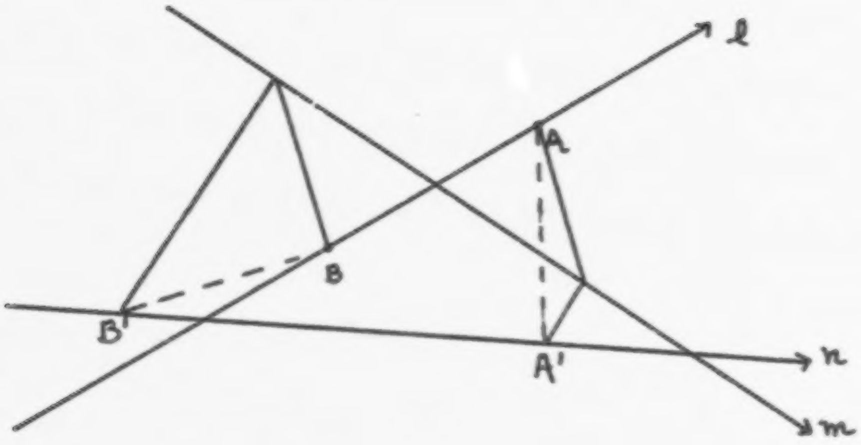
(b)



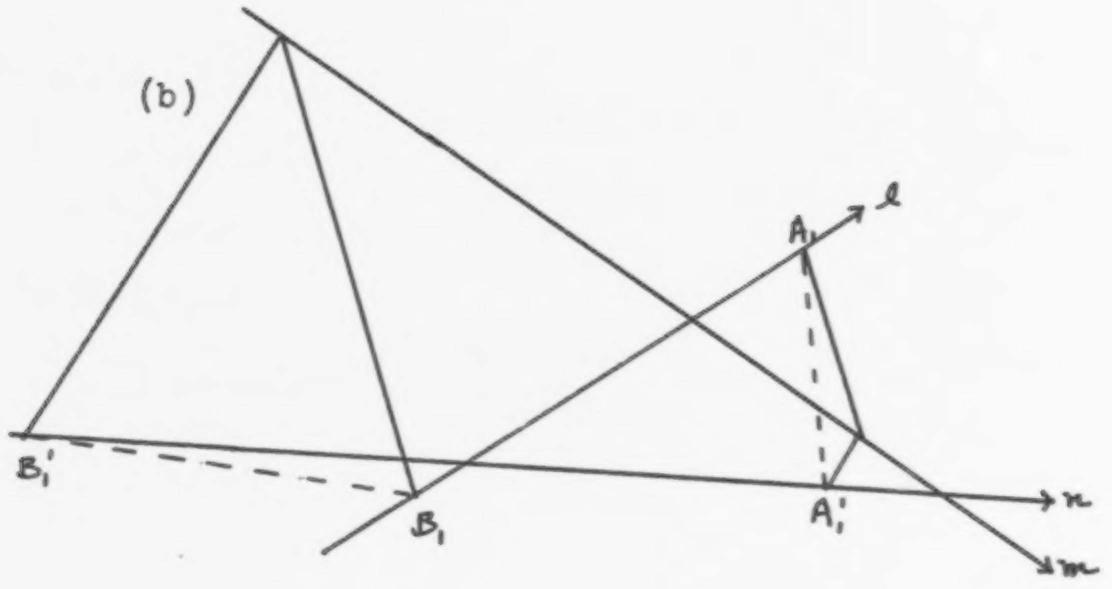
8.



(a)



(b)


l
A_1
B_1'
B_1
A_1'
n
m

Ans. No

3.14 Vectors (Time: 2 - 3 days)

In a sense, this section is a digression because it uses properties of a plane π which go beyond those prescribed in the axioms. It introduces the student to the notion of a vector, via the concept of "directed segments". In Course III these ideas will be defined precisely. For the present an intuitive approach via a physical model will be sufficient. In this connection a review of the notion of translation is helpful. Another helpful notion is that of a composition of translations, an idea which is closely related to addition of vectors.

Give the students lots of practice in drawing diagrams showing the various operations on vectors:

$$\vec{a}, \quad -\vec{a}, \quad \vec{a} + \vec{b}, \quad \vec{a} - \vec{b}$$

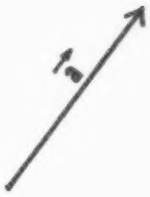
$$2\vec{a}, \quad 3\vec{a}, \quad (-2)\vec{a}$$

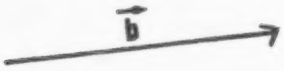
The arrow notation used here is temporary. Other notations for vectors will be introduced in Course III. Discuss non-mathematical interpretations of vectors such as forces, velocities, accelerations, price-vectors, etc. (See Kemeny, Finite Math, or Richardson, Fundamentals of Math).

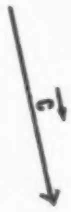
3.15 Exercises

These exercises provide graphical experiences with directed segments. Emphasize that the directed segments represent vectors. They are not themselves vectors. For

Exercise 3, please note: For convenience and easier checking of students' work, give students the following directed segments to use in doing problem 3.





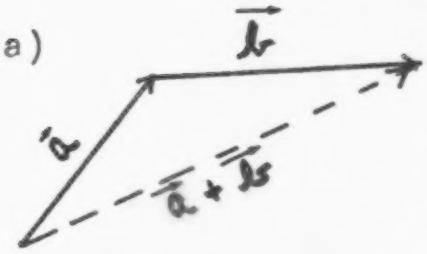


3.15 (Solutions to Exercises)

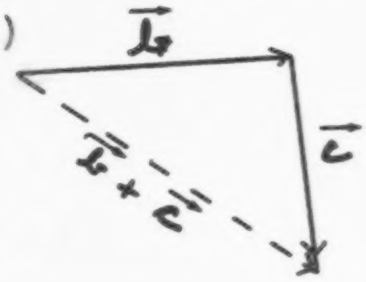
1. (a) one
   (b) two
   (c) two
   (d) two
   (e) three
2. (a) three
   (b) $\overrightarrow{AB} = \overrightarrow{ED}$
   $\overrightarrow{BC} = \overrightarrow{FE}$
   $\overrightarrow{CA} = \overrightarrow{DF}$

3.

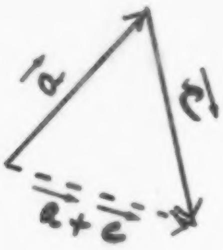
(a)


$\vec{b}$
$\vec{a}$
$\vec{a} + \vec{b}$

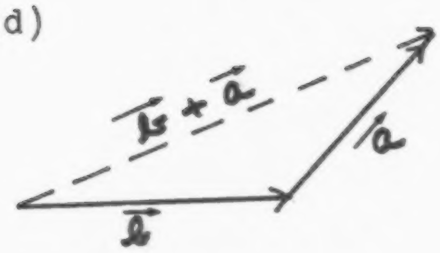
(b)


$\vec{b}$
$\vec{c}$
$\vec{b} + \vec{c}$

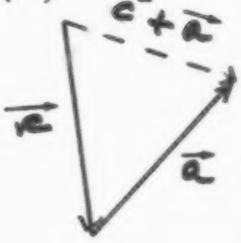
(c)


$\vec{a}$
$\vec{c}$
$\vec{a} + \vec{c}$

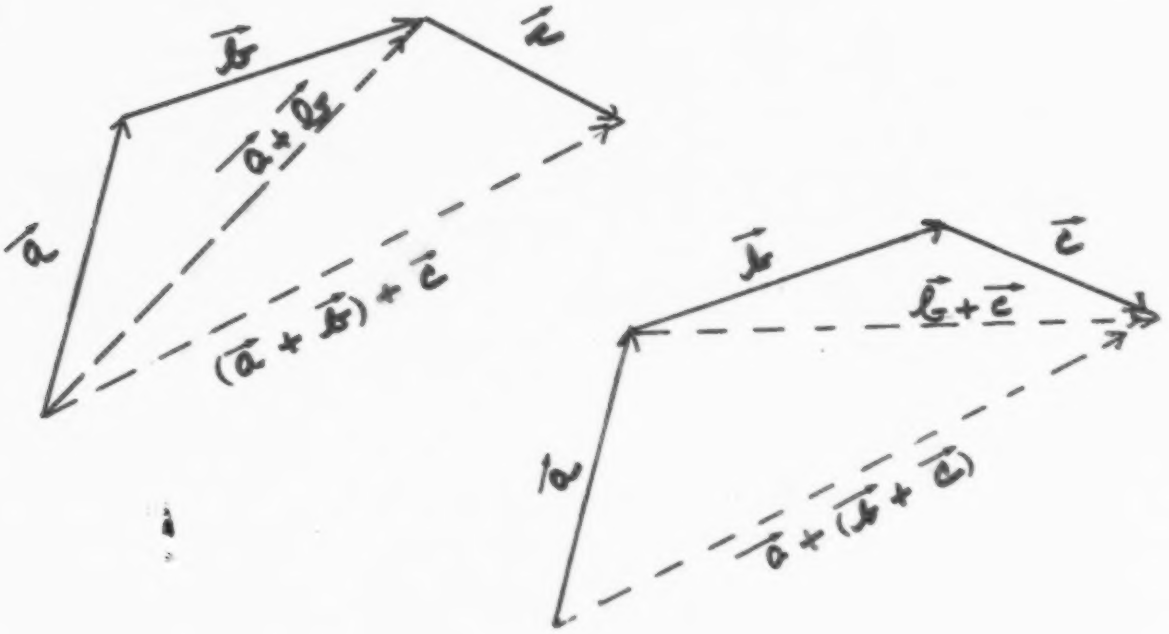
(d)


$\vec{b} + \vec{a}$
$\vec{a}$
$\vec{b}$

(e)


$\vec{c} + \vec{a}$
$\vec{c}$
$\vec{a}$

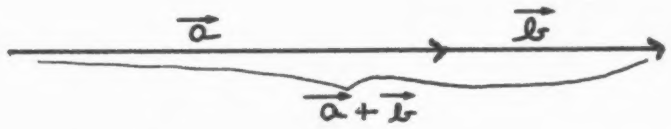
4.
(a)


$\vec{b}$
$\vec{c}$
$\vec{a} + \vec{b}$
$\vec{a}$
$(\vec{a} + \vec{b}) + \vec{c}$
$\vec{b}$
$\vec{c}$
$\vec{b} + \vec{c}$
$\vec{a}$
$\vec{a} + (\vec{b} + \vec{c})$

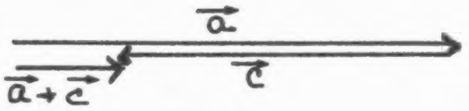
(b) The sum vectors appear to be the same, illustrating the ASSOCIATIVE LAW.

(c) Similar diagrams.

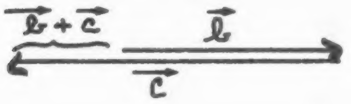
5. (a)


$\vec{a}$
$\vec{b}$
$\vec{a}+\vec{b}$

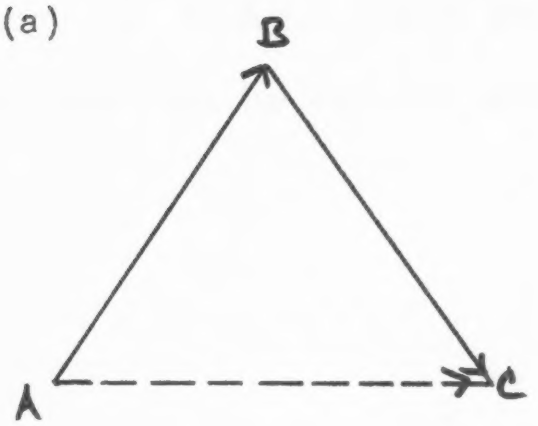
(b)


$\vec{a}$
$\vec{a}+\vec{c}$
$\vec{c}$

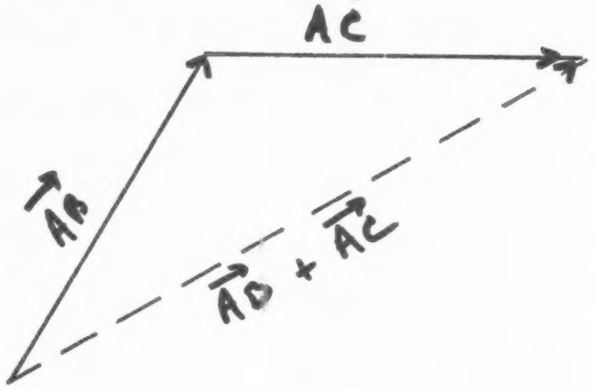
(c)


$\vec{b}+\vec{c}$
$\vec{b}$
$\vec{c}$

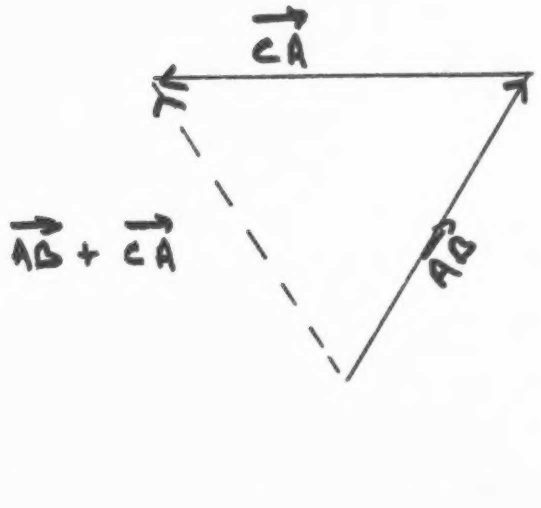
6. (a)



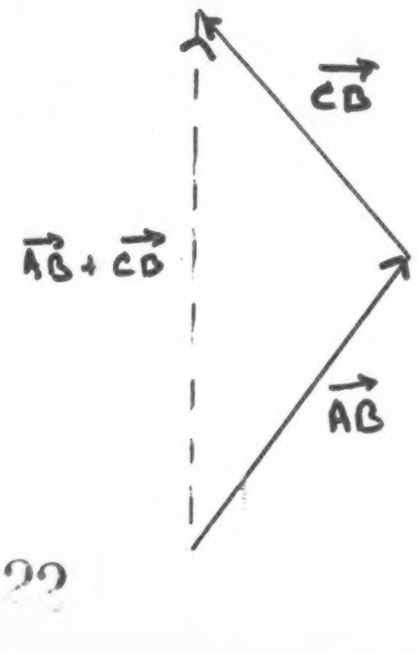
(b)


$\vec{AC}$
$\vec{AB}$
$\vec{AB}+\vec{AC}$

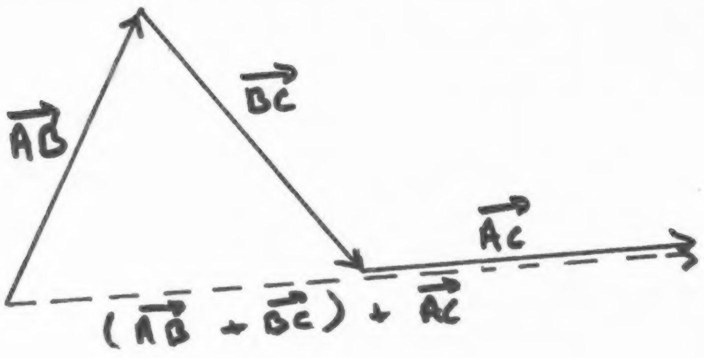
(c)


$\vec{CA}$
$\vec{AB}+\vec{CA}$
$\vec{AB}$

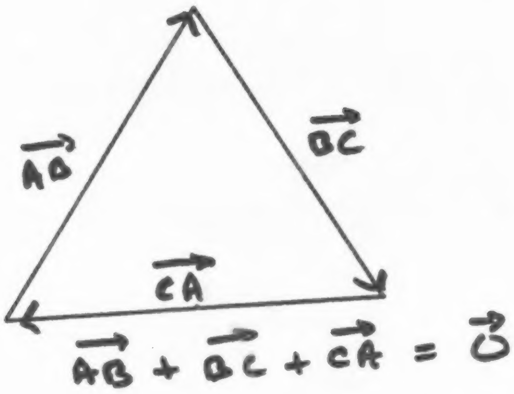
(d)


$\vec{CB}$
$\vec{AB}+\vec{CB}$
$\vec{AB}$

(e)

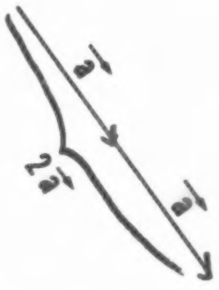

$\overrightarrow{AB}$
$\overrightarrow{BC}$
$\overrightarrow{AC}$
$(\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{AC}$

(f)

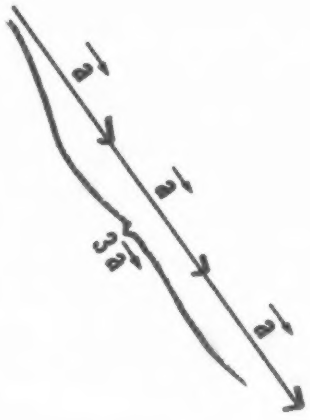

$\overrightarrow{AB}$
$\overrightarrow{BC}$
$\overrightarrow{CA}$
$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{O}$

The result in (f) introduces the concept of the zero vector, $\vec{0}$, and inverse vectors.

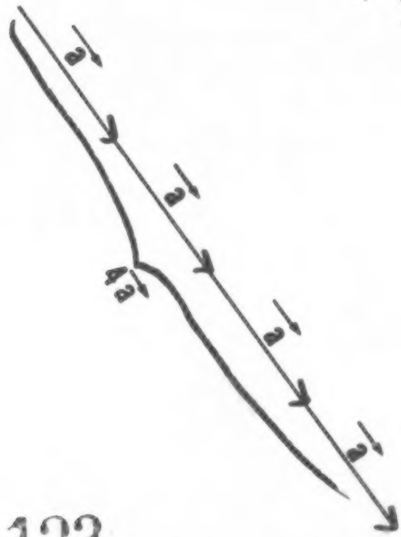
7. (a)


$\vec{a}$
$2\vec{a}$
$\vec{a}$

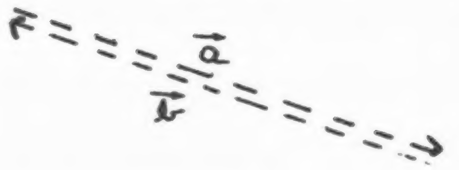
(b)


$\vec{a}$
$\vec{a}$
$3\vec{a}$
$\vec{a}$

(c)


$\vec{a}$
$\vec{a}$
$4\vec{a}$
$\vec{a}$
$\vec{a}$

8. (a)



$\vec{a} + \vec{b} = 0$

(b) $\vec{a} + \vec{o} = \vec{a}$ $\qquad$ $\vec{o} + \vec{a} = \vec{a}$

$\vec{b} + \vec{o} = \vec{b}$ $\qquad$ $\vec{o} + \vec{o} = \vec{o}$ $\qquad$ $\vec{o} + \vec{b} = \vec{b}$

General Rule: $\vec{o} + \vec{x} = \vec{x} + \vec{o} = \vec{x}$ for all vectors $\vec{x}$

9. By reasoning from properties of translations:
Since a vector $\vec{a}$ represents a translation there exists a unique inverse translation $\vec{x}$ which maps each point on to itself. The composition of these two translations is the zero translation: $\vec{a} + \vec{x} = \vec{x} + \vec{a} = \vec{0}$

3.16 Summary (Time: $2 - 2\frac{1}{2}$ days)

The axioms are listed here but not the theorems. However, each student may have built up a list of the theorems in his notebook.

3.17 Miscellaneous Exercises

Exercises 1, 2 and 3 are not difficult and provide a good review of concepts developed in this chapter. Exercise 4 is an exploratory one leading to generalizations in

Exercise 5. Proving these generalizations will probably be difficult for the average student. However the better students should find these results interesting and challenging. It is worth noting that there are still unsolved problems related to the ideas in Exercise 5.

Reference: Prenowitz; Jordan. Basic Concept of Geometry.

3.17 (Solutions to Miscellaneous Exercises)

1. By axiom 1a, plane π contains at least two (distinct) lines, call them m and n. By axiom 1b, line m contains at least two (distinct) points, call them A and B, and line n contains at least two (distinct) points, call them C and D. If C and D were both also in line m then by axiom 2, n would be the same line as m. Hence at least one of the points C and D, let us say C, is not in m. Therefore, A, B, C are three non-collinear points in P. By axiom 2 there is a line containing each pair of points {A, B}, {A, C}, and {B, C} we already know that m contains {A, B} and n contains {A, C} and that m and n are distinct lines. Let ℓ be the line containing {B, C}. ℓ must be distinct from either m or n for otherwise A, B and C would be collinear. Since ℓ already contains point B in m and point C in n, ℓ cannot contain any other point in either of these lines for otherwise by axiom 2 ℓ would be the same line as one of them. Therefore, lines ℓ, m, and n are non-concurrent.

2. (a) axioms 1 and 2 are true for this model but axiom 3 is false.

   (b) Three: {F, C}, {F, D} and {F, E}.

   (c) {A, C}, {A, D} and {A, E}

3. Referring to the proof of Theorem 13 (see Exercise 9 (a) in Section 3.7) we obtained four lines in π, m, n, ℓ and r and four points A, B, C, D such that:

   $m \parallel n$, $\ell \parallel R$

   A and B are in m

   C and D are in n

   A and D are in ℓ

   B and C are in r

   and no three of the four points are collinear. It follows that ℓ, m, n and r are distinct lines (otherwise the four points would be collinear). Moreover no three are concurrent because two out of any three of the lines are parallel.

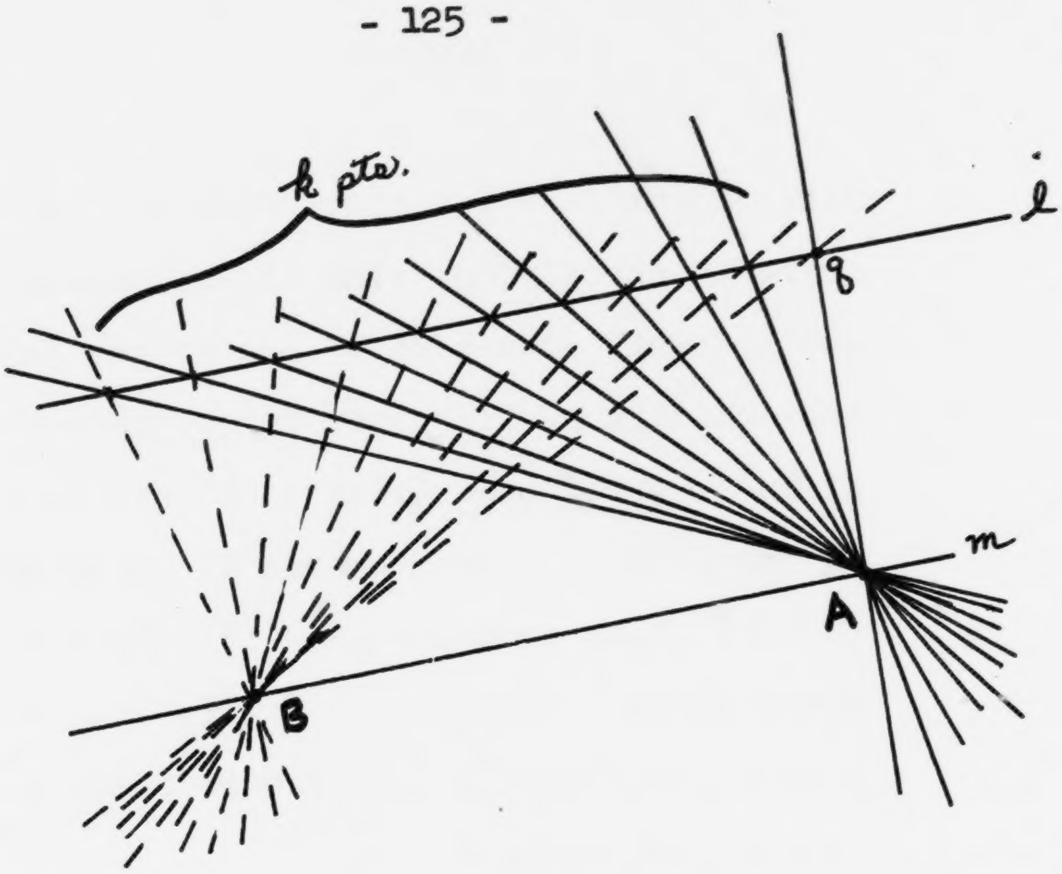
4. (a)

	No. of points in each line	No. of lines containing each point	No. of points in plane π	No. of lines in plane π
(4-pt. geom)	2	3	4	6
(9-pt. geom)	3	4	9	12
	4	5	16	20
(b)	k	k + 1	k^2	k(k + 1)

5. (a) Proof: (Using Theorem 11--See Exercise 9, Section 3.5)

Since one of the lines in plane P, call it line ℓ, contains exactly k points, and by theorem 1 there is a point A in P which is not in line ℓ, there exists a one-to-one correspondence between the set of all points in ℓ, and the set of all lines in P which contain A and are not parallel to ℓ. (by Theorem 11--see note above.) there exists, by Axiom 3, a line m containing point A and parallel to line ℓ. This line m must also contain another point B, by Axiom 2, and $B \notin \ell$. There also exists a one-to-one correspondence between the points in ℓ and the lines in P which contain B and are not parallel to ℓ. Now consider a line, q, containing point A and one of the k points contained in ℓ. By the same Theorem 11, there is a one-to-one correspondence between the set of points in q and the set of all lines in P which contain B and are not parallel to q. There are k such lines in P (the k lines containing B and the points of ℓ, minus the one line containing B and the chosen point of ℓ, plus the line m, containing B and A) and hence there must be k points on line q, and all lines of plane P.

Note: See figure next page



(b) Proof: (This follows from Theorem 11, also)

Let A be any point in P. By Theorem 4 there is a line m in P which does not contain A. By the result just proved in part (a), line m has exactly k points. By axiom 2 for each of these k points there is exactly one line in P containing that point and point A. In addition to these k lines each containing A and a point of m, there also exists in plane P (by axiom 3) exactly one line which contains A and is parallel to m. Since every line containing A is either parallel or else not parallel to m, this accounts for exactly (k + 1) lines in P containing A.

(c) By part (b) there are (k + 1) lines in plane P containing any given point A, in P. Let B be any other point in P (distinct from A). By axiom 2, there is exactly one line in P, containing both A and B. This line must be one of the $(k + 1)$ lines that contain A. In other words, each point B in plane P, other than A, is contained in exactly one of the $(k + 1)$ lines through A. Each of these $(k + 1)$ lines contains $(k - 1)$ points other than A. Hence plane P contains $(k + 1)(k - 1) = k^2 - 1$ points other than A. Therefore plane P contains k^2 points, including A.

(d) First Proof: Since each line in P contains exactly k points, (by part a) and each of these k points lies on exactly $k + 1$ lines (by part b), there are exactly $k \cdot (k + 1)$ lines in plane P.

Second Proof: Since plane P has k^2 points (by part c) there are $\frac{k^2 \cdot (k^2 - 1)}{2}$ pairs of distinct points in P. (To form all possible pairs of points, choose any one of the k^2 points for the first and any one of the remaining $(k^2 - 1)$ for the other. In the $k^2 \cdot (k^2 - 1)$ choices, each pair appears twice--as A, B and B, A for example. Hence, $\frac{k^2 \cdot (k^2 - 1)}{2}$ will be the number of pairs.) Each of these pairs determines a line in P, but these lines are not all distinct. In fact for every set of k points which are collinear there are exactly $\frac{k \cdot (k - 1)}{2}$ pairs of distinct points all of which determine the same line. (See reasoning above.) Hence the actual number of distinct lines in P is:

$$\frac{\frac{k^2(k^2 - 1)}{2}}{\frac{k(k - 1)}{2}} = k(k + 1)$$

Note: You will have to show students that

$k^2 - 1 = (k - 1)(k + 1)$ and hence $\frac{k^2 - 1}{k - 1}$
$= k + 1$

Sample Test Questions

I. True or False

(a) If line m is parallel to line n then m and n have no point in common. (m and n are not necessarily distinct.)

(b) In the plane π, if $m \parallel n$ and if line ℓ, in π, contains a point of m, then ℓ contains a point of n.

(c) Every model which satisfies axioms 1, 2, 3 must contain at least six lines.

(d) In every finite model which satisfies axioms 1, 2, 3 there are exactly the same number of lines containing any given point as there are points on any given line.

(e) If A is any point in plane π there are at least two lines in π which contain A and at least two lines in π which do not contain A.

II. If m and n are lines in π, under what circumstances would you say that

(a) m and n are parallel

(b) m and n are in the same equivalence class

III. Explain what is meant by each of the following:

(a) D_m (Assume the domain is the plane, π)

(b) E_m under D_m [alternate: $D_m(E)$]

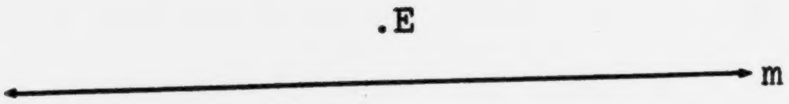
(c) $m \parallel n$

(d) aRb, if R is a relation

(e) $a \cap b = \emptyset$

IV. A proof of a simple theorem is given below with all the reasons omitted. There are 5 steps to the proof. On your own paper list the numbers 1 to 5 and supply the reason that fits the corresponding statement.

Theorem: In π if m is any line and E is any point not in m, then there are at least three lines containing E.



Proof:

(1) m has two points, call them "A" and "B".

(2) There is a line containing E and A, call it "r", and a line containing E and B, call it "s".

(3) There is a line containing E which is parallel to m, call it "t".

(4) Line t is distinct from lines r and s.

(5) Lines r and s ($\overleftrightarrow{EA}$ and $\overleftrightarrow{EB}$) are distinct.

V. Let R be a relation defined on all pairs of lines in π as follows:

x and y are in the relation R

if and only $x \cap y \neq \emptyset$

Decide whether or not R is an equivalence relation and explain. (Be liberal with partial credit for good thinking.)

VI. Relying only upon the axioms, prove the following:

Let m be a line in equivalence class D. Prove that there is a line distinct from m that is also in D. (Note: You may wish to let students rely upon the axioms and previously proven theorems.)

Answers to Sample Test Questions

I. (a) False (b) True (c) True (d) False (e) True

II. (a) Either m = n or $m \cap n = \emptyset$

(b) $m \parallel n$

III. (a) D_m is the parallel projection that maps plane π onto line m in the direction of D.

(b) E_m is the image of E in m under a parallel projection D_m.

(c) m and n are parallel, that is, either m = n or $n \cap m = \emptyset$

(d) $(a,b) \in R$, or a and b are in the relation R, or a is in the relation R to b.

(e) a and b have no points in common, or a and b are disjoint.

IV. 1. Every line in π contains at least two points. (Axiom 1b)

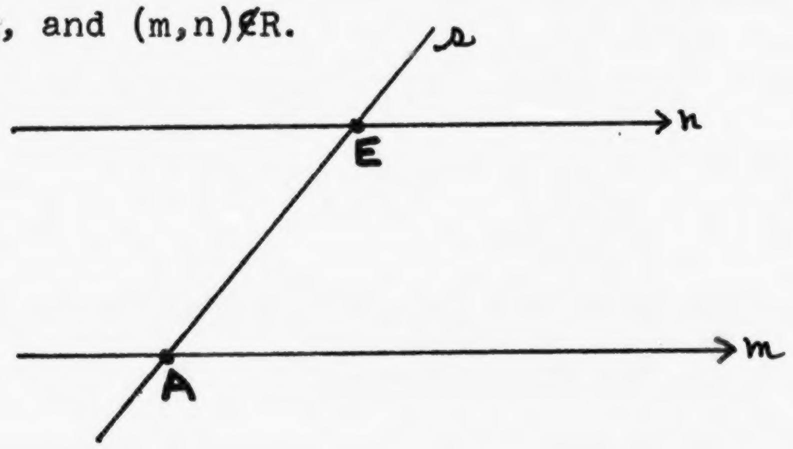
2. There is one and only one line containing any two points. (Axiom 2)

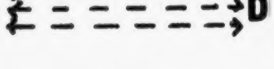
3. For every line m and point E there is one and only one line which contains E and is parallel to m. (Axiom 3)

4. r and s intersect m while t does not.

5. If EA and EB were the same line then points E, A, B would be in AB or m. As E is not in m, EA = EB.

V. R is not an equivalence relation because we could have distinct lines m, n, s such that $m \cap s \neq \emptyset$, $s \cap n \neq \emptyset$, and $m \cap n = \emptyset$. In other words, we could have $(m,s) \in R$, $(s,n) \in R$, and $(m,n) \notin R$.





VI. We have proved that for every line m there is a point, call it E, that is not in m. By axiom 3 there is a line, call it n, parallel to m and containing E. If E is not in m, $n \neq m$. But m and n are in the same equivalence class because $m \parallel n$.

Note: If you require that students rely only upon the axioms, they will have to prove that for every line m there is a point E not in m. (Theorem 1).

Chapter 4

Fields

Time estimate for Chapter: 14 days

General Comments:

This chapter has two major objectives --- to extend the study of abstract operational systems to fields and, in the process, to deepen student insight into the algebraic structure of the number systems. Since the additive and multiplicative structures of a field are groups (with a persistent problem involving o, the additive identity) the important theory of this chapter is that which concerns the interaction of the two operations via the distributive property.

The most interesting results are Theorems 5, 6, 11, 12, and 15.

Theorem 5. $a \cdot o = o \cdot a = o$

Theorem 6. $a \cdot b = o$ iff $a = o$ or $b = o$

Theorem 11. $(-a) \cdot b = a \cdot (-b) = -(a \cdot b)$

Theorem 12. $a \cdot (b - c) = a \cdot b - a \cdot c$

Theorem 15. $\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$

Since these are mostly familiar facts in $(Q, +, \cdot)$, it is important to use the finite fields $(Z_p, +, \cdot)$ in illustrations to show that the concepts have validity and usefulness in other settings.

Though the development of the theory of fields and ordered fields is rigorous --- in the sense of formally stated axioms and carefully proved theorems --- students should not be expected to memorize proofs to any particular theorems. There will be opportunities to discuss concepts of logic and proof, illustrating ideas from Chapter 1, but keep in mind the fact that almost all topics in the chapter will be developed further in future courses. The key sections of the present chapter are 4.4, 4.10, 4.12, 4.13, and 4.15. Sections 4.8 and 4.9 are not essential, although your class might find the use of fractions in finite fields fascinating.

The entire chapter should not take longer than 15 teaching days to cover (exclusive of Review Exercises and Chapter test.) The Review Exercises may be assigned during, at end, or in future spiralled assignments at the teacher's discretion.

4.1 What is a Field? (Time: 1 day)

The particular collection of properties used to define field is chosen to get a two-fold system that comes as near to being two interacting groups as possible and because three key number systems (two yet to come) have field structure. The non-symmetry of distributivity is forced by the behavior of o, the additive identity element, under multiplication. After the students have had experience with Theorems 4 and 6, it might be well to consider the following disproof which illustrates why addition does not distribute over multiplication.

$$1 + (1 \cdot o) = (1 + 1) \cdot (1 + o)$$

would imply $$1 + o = (1 + 1) \cdot 1$$

would imply $1 + o = 1 + 1$

would imply $o = 1$.

The need for different symbols for additive and multiplicative inverses is obvious: $-3 \neq 3^{-1}$. The choice of a^{-1} for multiplicative inverses might need some selling.

4.2 Exercises

1. Standard names

(a) Additive inverses

(1) $-\frac{2}{3}$
(2) $\frac{3}{4}$
(3) $-\frac{7}{8}$
(4) $\frac{1}{2}$
(5) $\frac{220}{3}$
(6) $\frac{31}{40}$
(7) $-\frac{28}{5}$
(8) -1
(9) 0

(b) Multiplicative inverses

(1) $\frac{3}{2}$
(2) $-\frac{4}{3}$
(3) $\frac{8}{7}$
(4) -2
(5) $-\frac{3}{220}$
(6) $-\frac{40}{31}$
(7) $\frac{5}{28}$
(8) 1
(9) none

2. Computations

(a) $\frac{11}{12}$
(b) $\frac{109}{96}$
(c) $\frac{11}{12}$
(d) $\frac{527}{288}$
(e) $-\frac{5}{4}$
(f) $-\frac{7}{360}$
(g) $-\frac{7}{360}$
(h) $-\frac{83}{240}$

3. Computations

(a) o
(b) o
(c) 1
(d) 1

4. Standard Names

(a) Additive inverses

(1) o
(2) 5
(3) 4
(4) 3
(5) 2
(6) 1

(b) Multiplicative inverses

(1) none
(2) 1
(3) none
(4) none
(5) none
(6) 5

5. Standard Names

(a) 4
(b) 2
(c) o
(d) 5
(e) 3
(f) none
(g) 4
(h) 3
(i) o

6. Fields?

(a) no inverses (There does not exist $a^{-1},^{-2},^{-3}, \ldots$ nor $\frac{1}{2}, \frac{1}{3}\ldots$)
(b) no · inverses
(c) no + inverses
(d) no + identity
(e) yes
(f) no · inverse for 2
(g) yes
(h) no · inverses for 2,3,4,6,8, 9,10
(i) no · inverses for 2,4,6
(j) no · inverses for 3,6
(k) no · inverses for 2,4,6,8,5

7. $x^2 = 2$, solvable in $(Z_7,+,\cdot)$ where $4^2 = 2$

8. (f), (h), (i), (j), (k)

9. Let a = p, and b = q. Then a · b = n which is o in $(Z_n,+,\cdot)$.

10. Yes. (Students should demonstrate the field properties which hold in this system.)

4.3 Getting Some Field Theorems Painlessly (Time: 2 - 3 days for 4.3, 4.4, 4.5)

Brush over this quickly, but point out the dividend of group theory and the translation of results from general to specific case.

4.4 Trouble with 0

There is no way, and perhaps expectedly so, to avoid the complications introduced when docile little 0 is asked to multiply. Emphasize the way that addition (0 is really only distinguishable as the additive identity), multiplication, and distributivity come together in Theorem 5 $(a \cdot 0 = 0)$. Stress that Theorems 5 and 6 can be restated compactly as follows:

$$\forall\ a, b \in F,\ a \cdot b = 0 \text{ iff } a = 0 \text{ or } b = 0$$

Although the solution of quadratic equations is considered in more detail in later sections (4.6, 4.13), the teacher may want to give several quadratic equations at this time as an immediate application of Theorems 5 and 6. For example:

$$x(x + 3) = 0$$

$$(x + 2)(x - \tfrac{1}{2}) = 0 \quad \text{etc.}$$

4.5 Exercises

1. Standard Names

(a) 1
(b) 10
(c) 3
(d) 3
(e) 3
(f) 5

2. Group Theorems

(a) For all a in S, $(a^I)^I = a$

(b) For all a,b in S, $(a * b)^I = b^I * a^I$

(c) For all a,b in S, $x * a = b$ has unique solution $x = b * a^I$.

3. $a \neq 0$ implies a^{-1} exists.

$\therefore a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c)$ Left operation which

implies $(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$ Associativity of $\cdot$

or $1 \cdot b = 1 \cdot c$ Definition of inverses

or $b = c$ Definition of identity

4. Standard names

(a) -21

(b) -21

(c) -21

(d) $-\frac{5}{32}$

(e) $\frac{1}{4}$

(f) $\frac{5}{9}$

5. Restatement of Theorem:

If $a \in (F,+,\cdot)$ and $a \neq 0$ then $a^2 \neq 0$.

Proof: (Indirect Method) Suppose $a^2 = 0$. We show that this leads to a contradiction.

$a^2 = 0$

$a \cdot a = 0$ Definition of a^2

$\therefore a = 0$ or $a = 0$ Theorem 6 $(a = b)$

But $a \neq 0$ by hypothesis.

Hence there is a contradiction and our supposition that $a^2 = 0$ must be false. Therefore $a^2 \neq 0$.

6. Computation

(a) 1

(b) 4

(c) 2

(d) 2

(e) 4

(f) 1

7. o

8. **Proof**: $a + b = a$ implies $a + b = a + o$

   implies $b = o$ by cancellation

   (Theorem 4.)

9. **Proof**: $(b + c) \cdot a = a \cdot (b + c)$ Commutativity for $\cdot$

   $= a \cdot b + a \cdot c$ Distributive Property

   $= b \cdot a + c \cdot a$ Commutativity for $\cdot$

10. If $n = p \cdot q$, there are elements a,b in Z_n and not equal to o such that $a \cdot b = o$. But by Theorem 6 this cannot happen in a field. Thus $(Z_n,+,\cdot)$ must not be a field.

4.6 Subtraction and Division (Time: 3 days for 4.6 and 4.7)

This section develops several important theorems and, therefore, time should be spend on this section so that the students will understand the proofs and applications to algebraic manipulation.

Examples 3 and 4 should be done in class as the students may have difficulty doing this on their own. Many more examples of this type should be done in class before doing section 4.7.

The teacher may wish to expand on this development by following section 4.7 by 4.13, 4.14, 4.15, 4.16. After completing the solution of quadratic equations, the teacher can return to 4.8 and complete the chapter.

4.7 Exercises

1. Standard Names

(a) $\frac{1}{3}$

(b) $-\frac{6}{5}$

(c) $\frac{7}{24}$

(d) $\frac{2}{7}$

(e) $\frac{27}{56}$

(f) $-\frac{5}{2}$

2. Standard Names

(a) 6

(b) 1

(c) 1

(d) 6

(e) 4

(f) 2

3. Simplifications

(a) $\frac{11}{6}x$

(d) $-\frac{1}{2}x$

(c) $\frac{20}{3}x$

(d) $14 + 15x$

4. Solution Sets

(a) $\{\frac{21}{25}\}$

(b) $\{0, \frac{2}{3}\}$

(c) $\{0, \frac{7}{4}\}$

(d) $\{0, \frac{4}{3}\}$

(e) $\{-3, \frac{1}{2}\}$

(f) $\{\frac{1}{6}, -\frac{5}{6}\}$

5. Proofs.

(b) $a - 0 = a + (-0)$ Definition of "-"

$= a + 0$ Definition of inverses

$= a$ Definition of Additive Identity

(c) $0 - a = 0 + (-a)$ Definition of "-"

$= -a$ Definition of Additive Identity

(d) $a - b = a - c$ implies $a + (-b) = a + (-c)$

Definition of "-"

implies $(-b) = (-c)$ Left Cancellation

implies $1(-b) = 1(-b)$ Left Operation

implies $(-1) \cdot b = (-1) \cdot c$ Theorem 11

implies $b = c$ Left Cancellation

(e)

$a - b = c$ implies $a + (-b) = c$ Definition of "-"

implies $a + (-b) + b = c + b$ Right Operation

implies $a + o = c + b$ Definition of Inverses

implies $a = c + b$ Definition of Additive Identity

$a = c + b$ implies $a + (-b) = c + b + (-b)$

Right Operation

$a + (-b) = c + o$ Definition of Inverses

$a + (-b) = c$ Definition of Additive Identity

$a - b = c$ Definition of "-"

6. (a) $a \div a = a \cdot a^{1}$ Definition of "÷"

$= 1$ Definition of inverses

(b) $a \div 1 = a \cdot 1^{-1}$ Definition of "÷"

$= a \cdot 1$ Definition of Inverse of Multiplicative Identity

$= a$ Definition of Multiplicative Identity

(c) $1 \div a = 1 \cdot a^{-1}$ Definition of "÷"

$= a^{-1}$ Definition of Multiplicative Identity

(d)

$a \div b = a \div c$ implies $a \cdot b^{-1} = a \cdot c^{-1}$

Definition of "÷"

$b^{-1} = c^{-1}$ Left Cancellation

$(b^{-1})^{-1} = (c^{-1})^{-1}$ Right Operation

$b = c$ Theorem 7

(e)

$a \div b = c$ implies $a \cdot b^{-1} = c$ Definition of "$\div$"

$a \cdot b^{-1} \cdot b = c \cdot b$ Right operation

$a = c \cdot b$ Definition of Inverses

7. Proof:

$a \cdot (b - c) = a \cdot (b + (-c))$ Definition of "-"

$= a \cdot b + a \cdot (-c)$ Distributivity

$= a \cdot b + -(a \cdot c)$ Theorem 11

$= a \cdot b - a \cdot c$ Definition of "-"

4.8 Fractions in Fields (Time: 1 day for 4.8 and 4.9)

This section affords an excellent opportunity to give students much needed computational practice with fractions. Therefore, even though this section is starred, it is recommended that time be devoted to exercises 4.9 either as a class lesson or through the homework. The teacher may also wish to supplement this section with additional problems involving fractions from a standard algebra text.

4.9 Exercises

1. Standard Names

(a) 5 (b) 5

(c) 2 (e) 3

(d) 2 (f) 3

2. Standard Names

(a) 2 (c) 4 (e) 4

(b) 2 (d) 4 (f) 4

3. Proof:

$\frac{d}{b} \cdot \frac{b}{d} = (d \cdot b^{-1}) \cdot (b \cdot d^{-1})$ Definition 4

$= (d \cdot d^{-1}) \cdot (b \cdot b^{-1})$ Associative and Commutative Property of Multiplication

$= 1 \cdot 1$ Definition of Inverses

$= 1$

4. Computations

(a) 4 (b) 4

5. (b) $-(\frac{-5}{2}) = -(\frac{2}{2}) = -1 = 6$

$\frac{5}{2} = 6$

(c) $\frac{-5}{-2} = \frac{2}{5} = 6$

$\frac{5}{2} = 6$

(d) $-(\frac{5}{2}) = -6 = 1$

$\frac{-5}{2} = \frac{2}{2} = 1$

6. Proof:

$-(\frac{-a}{b}) = -((-a) \cdot b^{-1})$ Definition 4

$= a \cdot b^{-1}$ Theorem 11 and Theorem 1

$= \frac{a}{b}$ Definition 4

7. Proof:

$-(\frac{a}{b}) = -(a \cdot b^{-1})$ Definition 4

$= (-a) \cdot b^{-1}$ Theorem 11(b)

$= \frac{-a}{b}$ Definition 4

4.10 Order in Fields (Time: 2 - 3 days for 4.10 and 4.11)

The questions of which familiar fields are orderable has been avoided intentionally (and deceitfully) until after some ordered field theory is developed because the rational number system is the ordered field that students have met. The real numbers, of course, will come in Chapter 5 and order will be the basic notion there.

In the present section the number of formal theorems has been limited to keep the topic manageable. The most important property in the section (in addition to the four basic axioms) is Theorem OF - 3(b)

$$a > b \text{ and } c < o \Rightarrow ac < bc.$$

The importance and use of this theorem should be illustrated in the solution of inequalities such as $-3x > 12$. Students will observe that except for OF - 3(b), there is a parallel between solution techniques for equations and inequalities. This will be used again in Section 4.13.

In the exercises of Section 4.11, numbers 9 and 10 should be covered; the results will be used later.

4.11 Exercises

Must do: numbers 9 and 10! (The iff indicates the reversibility of each step, so that two theorems are proved simultaneously.)

1. Correct symbols

(a) $>$ (b) $<$ (c) $>$ (d) $>$ (e) $>$

(f) $<$ (g) $>$ (h) $<$ (i) $>$ (j) $\geq$

(k) $\leq$

2. Equivalent inequalities

(a) $x > -\frac{11}{7}$ (b) $x > -8$ (c) $x^2 < 4$

or $-2 < x < 2$

3. Proof: $a < o$ iff $o < o - a$ Theorem 1 (a)

iff $o < -a$ o is the Additive Identity

4. $a < b$ iff $o < b - a$ Theorem 1 (a)

iff $o + -b < b + -a + -b$ OF 03

iff $-b < b + -b + -a$ Additive Identity and Commutative Property of +

iff $-b < o + -a$ Definition of Inverses

iff $-b < -a$ Additive Identity

5. Proof:

$$\left.\begin{array}{l} a > b \text{ implies } a + c > b + c \\ c > d \text{ implies } b + c > b + d \end{array}\right\} \text{ OF 03}$$

Thus $a + c > b + d$ by o - 1.

6. Proof:

$$\left.\begin{array}{l} a > o \text{ implies } a + b > o + b \\ b > o \text{ implies } o + b > o + o \end{array}\right\} \text{ OF 03}$$

Thus $a + b > o$ by o - 1.

7. Proof: $a > o$ and $b > o$ implies $a \cdot b > o \cdot b$ OF 04

implies $a \cdot b > o$ Theorem 5

8. Proof: (1) $a^2 = o$ implies $a = o$ Theorem 6

(2) $a < o$ iff $o < -a$ Theorem 1 (b)

$o < -a$ implies $o \cdot (-a) < (-a)(-a)$ OF 04

implies $o < a \cdot a = a^2$ Theorem 5 and Theorem 11 (c)

(3) $a > o$ implies $a \cdot a > a \cdot o$ OF 04

implies $a^2 > o$ Definition of $a \cdot a$ and Theorem 5

Note the proof here is by cases.

9. Find t so that $a = tb$, $o < a < b$ and $o < t < 1$.
   (a) For given a and b.
      (1) $t = \frac{1}{2}$ (2) $t = \frac{3}{5}$ (3) $t = \frac{33}{40}$
   (b) rule: $t = \frac{a}{b}$

10. Archimedean Property
   (a) Find n
      (1) $n \geq 25$
      (2) $n \geq 37001$
      (3) $n \geq 341$
   (b) Rule: $n > \frac{b}{a}$

11. For all x, $|x| = \begin{cases} x & \text{if } x \geq o \\ -x & \text{if } x < o \end{cases}$. This exhausts cases by o - 2

12. For all x, x is $\begin{cases} \text{positive} \\ \text{negative} \end{cases}$ iff $\begin{cases} x > o \\ x < o \end{cases}$. This exhausts cases by o - 2.

4.12 How Many Orderable Fields? (Time: $\frac{1}{2}$ day or independent reading assignment)

Although reaction to the non-orderable proof for $(Z_5,+,\cdot)$ might be a desire to throw out the axioms O1 - O4, in an operational system context it is useful to have an order relation that connects to the operations. Order as an independent concept can be and is defined somewhat differently in other situations.

4.13 Equations and Inequations in $(Q,+,\cdot,<)$ (Time: 2 days for 4.13 and 4.14)

As mentioned previously, this section can be done in conjunction with sections 4.6, 4.7, and 4.15. Examples 1 and 2 of this section illustrate in detail how the principles of logic and fields are used in the solving of equations. However, while students should be able to explain each step of an equation, they should not be expected to show all the details in their solution. You might want to give students more practice in solving equations than is contained in the exercises.

4.14 Exercises

1. Equivalent expressions

(a) $8x + (-7)$

(b) $-15x + \frac{5}{2}$

(c) $\frac{5}{3}x$

(d) $-6x^2 + 7\frac{1}{3}$

(e) $-\frac{31}{2}x + 60$

(f) $9x^2 + (-17)x$

2. Solution sets

(a) $\{-\frac{2}{3}\}$

(b) $\{-\frac{4}{15}\}$

(c) $\{\frac{36}{43}\}$

(d) $\{\frac{106}{9}\}$

(e) $\{\frac{35}{9}\}$

(f) $\{\frac{131}{36}\}$

(g) $\{\frac{37}{4}\}$

(h) $\{1\}$

3. Solutions of inequations

   (a) $x < \frac{3}{4}$ or solution set $S = \{x\colon x \in Q \text{ and } x < \frac{3}{4}\}$

   (b) $x > -\frac{5}{56}$ or solution set $S = \{x\colon x \in Q \text{ and } x > -\frac{5}{56}\}$

   (c) $x > -\frac{4}{3}$ etc.

   (d) $\frac{14}{45} < x$

   (e) $x < \frac{14}{9}$

   (f) $x < -\frac{2}{3}$

   (g) $x > \frac{36}{43}$

   (h) $\emptyset$

4. Solution sets

   (a) $\{28\}$ (b) $\{12\}$ (c) $\{8\}$ (d) $\{8\}$

5. 4737 cycles.

4.15 Solving Quadratic Equations (Time: 2 days for 4.15 and 4.16)

The first two sentences of this section will cause confusion. In the first place, students have indeed encountered equations involving symbols such as x^2 (See, for example, Section 4.7, ex. 4(b)). Furthermore, the second sentence gives the erroneous impression that equations involving symbols like x^2 are called linear equations. The difference between linear and quadratic equations should be made clear.

This section covers only quadratic equations that are easily factorable over the rationals. More general techniques appear in Course III, Chapter 5 on "Polynomials."

Again you may want to supplement the exercises on factoring and equation solution to meet the needs of your class. Remember, however, the topics will be touched again in a future course.

4.16 **Exercises**

1. **Products**

(a) $x^2 + 18x + 77$

(b) $x^2 + \frac{1}{8}x - \frac{5}{16}$

(c) $x^2 - 6x + 9$

(d) $x^2 + 14x - 176$

(e) $x^2 - 14x - 176$

(f) $x^2 - 30x + 176$

(g) $x^2 - 9$

(h) $x^2 + x - 20$

(i) $12x^2 + 23x + 10$

(j) $\frac{10}{7}x^2 - \frac{290}{21} + 10$

2. **Factored form**

(a) $(x + 5)(x + 4)$

(b) $(x - 5)(x - 4)$

(c) $(x + 5)(x - 4)$

(d) $(x - 5)(x + 4)$

(e) $(x - 10)(x + 2)$

(f) $(x + 2)(x + 10)$

3. **Solution sets**

(a) $\{0, 11\}$

(b) $\{\frac{2}{3}, -\frac{4}{3}\}$

(c) $\{0, -\frac{3}{4}\}$

(d) $\{\pm \frac{4}{5}\}$

(e) $\{\pm 8\}$

(f) $\{0, -\frac{13}{3}\}$

4. **Solution sets**

(a) $\{-3, -5\}$

(b) $\{4, 2\}$

(c) $\{-13, 2\}$

(d) $\{-3\}$

(e) $\{3\}$

(f) $\{6, -2\}$

(g) $\{-6, 2\}$

(h) $\{4, 3\}$

5. **Solution sets**

(a) $\{\frac{19}{3}\}$

(b) $\{0, \frac{19}{3}\}$

(c) $\{\frac{7}{5}\}$

(d) $\{-3\}$

(e) $\{5\}$

6. Solution sets

(a) $-2 < x < -1$ (c) $-5 < x < 5$

(b) $x < -7$ or $x > 2$ (d) $\frac{1}{3} < x < 1$

7. Solution sets

(a) $\{2, 9\}$ (c) $\{3, 5\}$

(b) $\{9, 10\}$ (d) $\{5, 6\}$

Answers to questions in Note: Yes, 6 No

8. Solution sets

(a) $\{-4, -\frac{2}{3}\}$ (c) $\{-\frac{1}{2}, -1\}$

(b) $\{2, -\frac{1}{4}\}$ (d) $\{\pm\frac{5}{3}\}$

4.18 Review Exercises

1. Computations

(a) 6 (b) 3 (c) 0 (d) 5

2. Computations

(a) $-\frac{10}{21}$ (c) $\frac{760}{9}$

(b) $\frac{7}{75}$ (d) $-\frac{15}{14}$

3. Proofs

(a) $-(-a) = a \therefore -(-(-a)) = -a$ by S P E

(b) $a \cdot b = a \cdot c$ implies $a^{-1} \cdot a \cdot b = a^{-1} \cdot a \cdot c$ Left Operation

implies $1 \cdot b = 1 \cdot c$ Definition of Inverses

implies $b = c$ Definition of Multiplicative Identity

(c) $(x - a)(x - b) = (x + (-a))(x + (-b))$ Definition of "-"

$= (x + (-a))x + (x + (-a))(-b)$

Distributive Property

$= x^2 + (-a)x + (-b)x + (-a)(-b)$

Distributive Property

$= x^2 - ax - bx + ab$

Theorem 11(a), (c), and Definition of "-"

(d) follows from (c) when $b = a$.

4. Proofs:

(a) $a < b$ implies $a + b < 2b$

implies $\frac{a + b}{2} < b$

(b) $a < o$ and $b > o$ implies $ab < b \cdot o$

implies $ab < o$

(c) Note: Before assigning this exercise, you should prove that $1 > 0$ in an ordered field.

Indirect Proof: Suppose $\frac{1}{a} \not> 0$, then by Trichotemy property either $\frac{1}{a} = 0$, or $\frac{1}{a} < 0$. Since o has no multiplicative inverse in $(F, \cdot)$ $\frac{1}{a} \neq 0$. Suppose $\frac{1}{a} < 0$, then by 4(b) above, $a \cdot \frac{1}{a} < 0$ which implies $1 < 0$. This is a contradiction. Therefore $\frac{1}{a} > 0$.

5. Solution sets

(a)	$\{ \frac{340}{7} \}$	(d)	$\{0, \frac{7}{3}\}$
(b)	$\{\frac{12}{7}\}$	(e)	$\{-8, -9\}$
(c)	$\{\pm \frac{7}{12} \}$	(f)	$\{-3, -2\}$

Suggested Chapter Test

1. Determine which of the following statements are true and which are false where a, b, c, d are any elements of a field.

   (a) $a \cdot b = 0$ implies $a = 0$.

   (b) $a + (b \cdot c) = (a + b) \cdot (a + c)$.

   (c) $-(a \cdot b) = a \cdot (-b)$.

   (d) $1 \neq 0$.

   (e) $-(a - b) = -a + b$.

   (f) $a \cdot b = a \cdot c$ implies $b = c$.

2. Determine which of the following statements are true and which are false where a, b, c, d are any elements of an ordered field.

   (a) $a^2 \geq 0$

   (b) $-a < a$

   (c) $a > b$ implies $ac > bc$

   (d) $a < b$ and $a < c$ implies $b < c$

3. Calculate standard names for each of the following in $(Z_{11},+,\cdot)$.

   (a) 9^{-1}

   (b) $-(7 - 8)$

   (c) $(3 \cdot 2^{-1})^{-1}$

   (d) -8

   (e) $-(8^{-1})$

   (f) $5^{-1} \cdot (7 - 10)$

4. Prove: For all a, b, c in field $(F,+,\cdot)$ if $c \neq 0$, then $(a + b) \div c = (a \div c) + (b \div c)$.

5. Prove: For all a, b, c in ordered field $(F,+,\cdot,<)$, if $a < b$ then there is $c > o$ in F such that $a + c = b$.

6. Find the solution set of each of the following open

sentences where the domain of "x" is (Q,+,·).

(a) $7(x + 2) = 8x = 3$

(b) $7x^2 - 3x = 0$

(c) $4x^2 - 9 = 0$

(d) $x^2 + 11x + 24 = 0$

(e) $x^2 - 5x - 4 = 0$

(f) $7x + 5 < 3 - 4x$

Answer Key for Chapter Test

1. (a) False

(b) False

(c) True

(d) True

(e) True

(f) False (This would be true iff $a \neq 0$).

2. (a) True

(b) False

(c) False

(d) False

3. (a) 5

(b) 1

(c) 8

(d) 3

(e) 4

(f) 6

4. $a, b, c \in (F,+,\cdot)$ and $c \neq 0$

$(a + b) \div c \Rightarrow (a + b) \cdot c^{-1}$ Definition of "$\div$"

$\Rightarrow (c^{-1}) \cdot (a + b)$ Commutativity for $\cdot$

$\Rightarrow (c^{-1} \cdot a) + (c^{-1} \cdot b)$ Distributive Property of $\cdot$ over $+$

$\Rightarrow (a \cdot c^{-1}) + (b \cdot c^{-1})$ Commutativity for

$\Rightarrow (a \div c) + (b \div c)$ Definition of division

5. $\forall a, b, c \in (F,+,\cdot,<)$ and $a < b$

$a < b \Rightarrow a + (-a) < b + (-a)$ Right Operation

$\Rightarrow 0 < b + (-a)$ Definition of Inverses

Let $c = b + (-a)$

Then $a + c = a + [b + (-a)]$ S P E

$= a + [(-a) + b]$ Commutativity for +

$= [a + (-a)] + b$ Associativity for +

$= 0 + b$ Definition of inverses

$= b$ Additive Identity

Then $a + c = b$

6. (a) $\{17\}$

(b) $\{0, \frac{3}{7}\}$

(c) $\{\pm \frac{3}{2}\}$

(d) $\{-3, -8\}$

(e) $\{1, 4\}$

(f) $\{x\colon\ x \in Q \text{ and } x < -\frac{2}{11}\}$

Chapter 5

The Real Numbers

Time Estimate for chapter: 14 days

Part I of this chapter is designed to motivate the need to extend the ordered field of rational numbers $(Q, +, \cdot, <)$ to the complete ordered field of real numbers $(R, +, \cdot, <)$. The motivation is essentially geometric; that is, $(Q, +, \cdot, <)$ is shown to be inadequate to express the length of every line segment. In order to overcome this inadequacy, the measuring process is examined and is seen to produce a set of rational numbers, each of which may be viewed as an approximation to the length of the segment being measured. (One of these numbers may, indeed, be the length.) Two important points emerge from the general discussion:

1) The length of a line segment is the least upper bound of the set of rational numbers which arises from the measuring process.
2) Some sets of rational numbers arising from the measuring process do not have rational least upper bounds; therefore, $(R, +, \cdot, <)$ is introduced as an extension of $(Q, +, \cdot, <)$ so that every set of rational numbers produced by the measuring process will have a least upper bound. In fact, the real number system $(R, +, \cdot, <)$ is characterized more generally by the Completeness Property:

> Every non-empty set of real numbers which is bounded from above has a least upper bound.

5.1 (Time: 2 days)

This section provides an algebraic example, illustrating the inadequacy of $(Q, +, \cdot)$ to solve all the equations we might encounter. The proof that the solution set of $\underline{x^2 = 2}$ is empty in $(Q, +, \cdot)$ depends upon the Unique Factorization Property (UFP) of the natural numbers. Since this property was last covered in Chapter 11 of Course I, a chapter which was read independently by many students, it will have to be reviewed at this time. Have students speculate about how the number of factors of a given prime p contained in the complete factorization of n is related to the number of factors of p contained in the complete factorization of n^2. Include the possibility that n contains zero factors of p. Some students might be encouraged to read the more standard proof that $\sqrt{2}$ is irrational which appears in last year's experimental edition of Course II.

The number of exercises in this chapter is absolutely minimal and most students should be expected to try each one.

5.2 Answers to Exercises

1. (a) $20 = 2^2 \cdot 5$; 0 factors of 3

    $(20)^2 = 2^4 \cdot 5^2$; 0 factors of 3

    (b) $42 = 2 \cdot 3 \cdot 7$; 1 factor of 3

    $(42)^2 = 2^2 \cdot 3^2 \cdot 7^2$; 2 factors of 3

    (c) $2250 = 2 \cdot 3^2 \cdot 5^3$; 2 factors of 3

    $(2250)^2 = 2^2 \cdot 3^4 \cdot 5^6$; 4 factors of 3

    (d) $270 = 2 \cdot 3^3 \cdot 5$; 3 factors of 3

    $(270)^2 = 2^2 \cdot 3^6 \cdot 5^2$; 6 factors of 3

(e) $891 = 3^4 \cdot 11$;   4 factors of 3

$(891)^2 = 3^8 \cdot 11^2$;   8 factors of 3

There are twice as many factors of 3 in the complete factorization into primes of n^2 as in n. Thus, n^2 must contain an even number of factors of 3. This applies not only to 3, but to any prime p.

2. (a) i. Suppose $(\frac{p}{q})^2 = 3$, $p, q \in Z$, $q \neq 0$. Then $\frac{p^2}{q^2} = 3$ and $p^2 = 3q^2$. p^2 contains an even number (possibly 0) of factors of 3 in its complete factorization into primes. Likewise, q^2 contains an even number (possibly 0) of factors of 3. Thus, $3q^2$ contains an odd number of factors of 3. By the Unique Factorization Property, we cannot have $p^2 = 3q^2$. Thus, there is no rational number whose square is 3.

ii. Suppose $(\frac{p}{q})^2 = 5$, p and $q \in Z$, $q \neq 0$. Then $\frac{p^2}{q^2} = 5$ or $p^2 = 5q^2$. p^2 contains an even number (possibly 0) of factors of 5 in its complete factorization into primes. Likewise, q^2 contains an even number (possibly 0) of factors of 5 in its complete factorization. Then $5q^2$ contains an odd number of factors of 5. By the UFP, we cannot have $p^2 = 5q^2$. Thus, there is no rational number whose square is 5.

iii. Suppose $(\frac{p}{q})^2 = 6$, $p, q \in Z$, $q \neq 0$. Then $\frac{p^2}{q^2} = 6$ or $p^2 = 6q^2 = 2 \cdot 3q^2$. Apply the same reasoning as in (i) above.

(b) Suppose $(\frac{p}{q})^2 = 4$ $p, q \in Z$, $q \neq 0$

then $\frac{p^2}{q^2} = 4$

then $p^2 = 4q^2 = 2^2q^2$

p^2 contains an even number of factors of 2.

q^2 contains an even number of factors of 2.

Thus 2^2q^2 contains an even number of factors of 2.

Thus $p^2 = 4q^2$. Therefore the solution set of $x^2 = 4$ is in $(Q, +, \cdot)$

3. (a) $\{+5, -5\}$ (b) $\{+5, -5\}$ (c) $\{+15, -15\}$ (d) $\{0\}$
   (e) $\emptyset$

4. (a) $\{1.4, 1.41\}$ (b) $\{1.41, 1.414\}$ (c) $\{1.73, 1.732\}$
   (d) $\{1.73, 1.732\}$

5.3 (Time: 1 day)

This section relies on the Pythagorean property which many students are seeing for the first time. They should be reassured that this relationship will be proved in a subsequent chapter. In this discussion, it becomes apparent that $(Q, +, \cdot)$ is not adequate to satisfy all our geometrical needs; that is, we cannot express the length of every line segment with a rational number once a unit length has been selected. The importance of a unit length must be stressed. For example, we might take the diagonal in question and represent that by one unit. Thus, we would have a rational number to express its length. However, in terms of this chosen unit, the length of a diagonal of a new square whose side has this unit length would then not be a rational number.

The discussion of the measuring process may be difficult to understand with the assorted number of diagrams and possibilities included. The ideas are easy; in fact, they are much easier to express orally than in writing. A set of rational numbers emerges from measurement; this set may be finite or infinite. Let students speculate about the consequences of these possibilities. Remember that if the process ends, the length is a rational number but that the converse of this statement is not true. A segment may have a rational length ($\frac{1}{3}$) while the measuring process applied to the segment may produce an infinite set ($\{0, .3, .33, .333, \ldots\}$). The student cannot really see, at this point, that the measuring process may not end. A possible explanation is that if the process does end, the length must be a rational number. However, we already know that certain segments do not have rational lengths.

5.4 Answers to Exercises

1. (a) $\{6, 6.1\}$
   6 segments each of length 1 cm.
   1 segment of length $\frac{1}{10}$ cm.

   (b) $\{0, 0.3, 0.32\}$
   0 segments of length 1 cm.
   3 segments each of length $\frac{1}{10}$ cm.
   2 segments each of length $\frac{1}{100}$ cm.

   (c) $\{47, 47.5, 47.50, 47.503\}$
   47 segments each of length 1 cm.
   5 segments each of length $\frac{1}{10}$ cm.
   0 segments of length $\frac{1}{100}$ cm.
   3 segments each of length $\frac{1}{1000}$ cm.

   (d) $\{2, 2.1, 2.15, 2.153, 2.1539, 2.15398\}$
   2 segments each of length 1 cm.
   1 segment of length $\frac{1}{10}$ cm.

5 segments each of length $\frac{1}{100}$ cm.

3 segments each of length $\frac{1}{1000}$ cm.

9 segments each of length $\frac{1}{10,000}$ cm.

8 segments each of length $\frac{1}{100,000}$ cm.

2. (a) 3.728 or any other rational number greater than 3.728.
   (b) 1 or any rational number greater than 1.
   (c) 1.72 or any rational number x so that $x \geq \frac{170}{99}$.
   (d) Any rational number.

3. (a) 3.728 (b) 1 (c) $1\frac{71}{99}$ (d) None exists at this time

4. (a) $a = 4$ (b) $b = 24$ (c) $c = 25$ (d) None
   (e) $a = 8$

5. (a) $\{1, 2, 3\}$ (b) $\{1, 2, 3, \ldots 33\}$ (c) $\{1, 2, 3\}$
   (d) $\{1, 2, 3\}$

5.5 (Time: 2 days)

This section is extremely important. It introduces the following definitions:

1. upper bound
2. least upper bound
3. length of a line segment

Several examples of finite sets which arise from the measuring process should be presented. In each case, the length is known to be the largest rational number in the set. This number is clearly the least upper bound of the set. Formal proofs for this fact, though very easy, should not be presented. At this time, we should not complicate something that students will see very easily.

The important thing is to make a strong case for the least upper bound as being a reasonable candidate for the length of a line segment. You will have to rely on the diagrams and discussion of the measuring process to get this across. You might ask students to speculate about the existence of a least upper bound in $(Q, +, \cdot. <)$ for each set of rational numbers arising from the measuring process. The case of the diagonal of a square discussed in Section 5.3 might allow some students to see that a rational l.u.b. does not always exist. What do they suggest be done to overcome this difficulty?

5.6 Answers to Exercises

1. (a) Any rational number x so that $x \geq 4$
   (b) Any rational number
   (c) Any rational number x so that $x \geq 1.9$
   (d) There is no upper bound
   (e) $\{x \in Q : x > 3\}$
   (f) $\{x \in Q : x \geq 1\}$

2. (a) 4 (b) There is none (c) 1.9 (d) None
   (e) Cannot say (f) None exists

3. (a) 2 (b) $\frac{1}{3}$ (c) 7.145 (d) 1.6668

4. (a) If A is bounded from above, there is an $r \in Q$ so that $x \leq r$ for each $x \in A$. Likewise, there is an $r' \in Q$ so that $x \leq r'$ for each $x \in B$. Let $p = \max(r, r')$. Then, for each $a \in A \cup B$, $a \leq p$. Thus, p is an upper bound for $A \cup B$.
   (b) Since A is bounded from above, there is an $r \in Q$ so that

$x \leq r$ for each $x \in A$. But, if $y \in A \cap B$, then $y \in A$ and $y \leq r$. Thus, r is an upper bound of $A \cap B$. Notice that the boundedness of B is not needed.

5. Let $a \in A$. Since x is an upper bound of A, $a \leq x$. But $x < y$. Thus, by transitivity of "$<$", we have $a < y$. Thus, y is an upper bound of A.

6. Since x and y are both least upper bounds, each is an upper bound. By definition of x being a l.u.b., $x \leq y$. By definition of y being a l.u.b., $y \leq x$. Thus, $x = y$.

5.7 (Time: 1 day)

This section illustrates the various significant cases which might arise from the measuring process. The first case is very easy and the student is probably convinced already that in the finite case, the least upper bound is clearly the length. We must honestly admit that we do not need any special process (l.u.b.) to find the length of a line segment in the event that the measuring process does end. The least upper bound is really needed in the event that the measuring process produces an infinite set of rational numbers, each approximating the length in question. The sketch of the proof that $\frac{1}{3}$ is the l.u.b. of $\{0, 0.3, 0.33, \ldots\}$ is a good introduction to a proof by induction. In fact, the reason that this demonstration is called a sketch rather than a proof is that a complete induction argument is not presented. The sketch will be difficult for most students.

The student sees much more clearly now that for certain sets of rational numbers arising from the measuring process, there are no rational least upper bounds. Ask the student again to guess what can be done to overcome this problem. Also ask him why we want each of these sets to have a least upper bound.

5.8 Answers to Exercises

1. (a) .3 (b) .33 (c) .333 (d) $\frac{1}{3}$

2. (a) $\{1, 2, 3, 4, 5, 6\}$ (b) $\{1, 2, 3, \ldots 66\}$
   (c) $\{1, 2, 3, 4, 5, 6\}$ (d) $\{1, 2, 3, 4, 5, 6\}$

3. 0, 0.6, 0.66

4. The length is a rational number. If the measuring process produced the set $\{k, k \cdot a_1, k \cdot a_1 a_2, \ldots k \cdot a_1 a_2 \ldots a_n\}$, then the least upper bound would be $k \cdot a_1 a_2, \ldots a_n$ which is a rational number.

5. (a) .6 (b) $.142857\overline{142857}$ (c) $.22\overline{2}$ (d) .475
   (e) .125

5.9 (Time: 2 days)

The real number system is introduced, in this development, so that we can measure the length of every line segment. Thus, every set of rational numbers arising from the measuring process must have a least upper bound in $(R, +, \cdot, <)$. This conclusion has been reasonably motivated in the previous sections of the chapter and should not cause too much trouble for the student. However, we jump to a more general, more powerful statement rather quickly, as expressed by the Completeness Property:

> Every non-empty set of real numbers which is bounded from above has a least upper bound.

The Completeness Property is one of the essential differences between the rationals and the reals and should receive considerable emphasis. Do not expect students to grasp this quickly and they, themselves, should be told that this property, though seemingly simple in statement, expresses one of the most difficult mathematical concepts. This is just a first introduction to the notion of completeness; do not attempt to exhaust it.

5.10 Answers to Exercises

1. (a) W, Z, Q, R (b) W, Z, Q (c) W, Z, R (d) W, Z, Q, R

2. (a) irrational (b) rational (c) neither (d) rational
(e) rational (f) irrational

3. Assume $a \cdot b$ is a rational number; that is, there are integers p, q where $q \neq 0$ and $a \cdot b = \frac{p}{q}$.
Since b is a non-zero rational number, there are integers e and f where $e \neq 0$ and $f \neq 0$ and $b = \frac{e}{f}$.
Thus, $a \cdot \frac{e}{f} = \frac{p}{q}$ or $a = \frac{p \cdot f}{e \cdot q}$
Since $e \neq 0$ and $q \neq 0$, then $e \cdot q \neq 0$. This means that a is a rational number. However, we were given that a is an irrational number. Thus, our assumption is false and we conclude that $a \cdot b$ is an irrational number.

4. Consider $x \in R$ where $x^2 = 2$.
x is irrational, but $x \cdot x = x^2 = 2$. Thus $x \cdot x$ is a rational number. Also consider $x^2 = 2$, $y^2 = 8 \Rightarrow x^2y^2 = 16$, $(xy)^2 = 16$, $xy = 4$

5. If x is a rational number, there are integers p and q where

$q \neq 0$ and $x = \frac{p}{q}$. If y is a rational number, there are integers a and b where $b \neq 0$ and $y = \frac{a}{b}$. Thus, $x \cdot y = \frac{p}{q} \cdot \frac{a}{b} = \frac{p \cdot a}{q \cdot b}$. Since $q \neq 0$ and $b \neq 0$, then $q \cdot b \neq 0$. We see that $x \cdot y$ is a rational number.

6. (a) $.4139$ (b) $.338488\overline{8}$ (c) the numbers are equal
   (d) $.3644\overline{4}$ (e) the numbers are equal

7. (a), (c), (e).

5.11 (Time: 2 days)

In general, two real numbers can be ordered in terms of their infinite decimal representations in the usual way, by comparing them digit by digit until they disagree. The modification of this procedure for certain rational numbers is presented in this section but not discussed at length. The teacher should be aware that this modification is meant to accomodate rational number decimal representations which have a bar "-" above a zero. ($.230\overline{0}$). In such a case, there are two distinct decimal representations for the same number ($.230\overline{0}$ and $.229\overline{9}$). Since this is characteristic only of certain rational numbers, perhaps it should have been pointed out in Course I, Chapter 12 on the rational numbers. Students should not think that the extension from $(Q, +, \cdot)$ to $(R, +, \cdot)$ necessitated this modification.

Because of the reordering of chapters, the definition of the square root of a number and the symbol "$\sqrt{\ }$" appear for the first time in this chapter. Unfortunately, this complicates matters somewhat but consider the following example as representative of what we are trying to get across.

1) $\ell = \sqrt{2}$ if and only if $\ell^2 = 2$.

4. Assume $4 + 3\sqrt{2}$ is a rational number; that is, say there are integers p and q where $q \neq 0$ and $4 + 3\sqrt{2} = \frac{p}{q}$.
Then, $3\sqrt{2} = \frac{p}{q} - 4 = \frac{p - 4q}{q}$ or $\sqrt{2} = \frac{p - 4q}{3q}$.
But $p - 4q$ and $3q$ are integers; since $q \neq 0$, then $3q \neq 0$.
This means that $\sqrt{2}$ is rational which we know is not true.
Thus, the assumption is false and we conclude that $4 + 3\sqrt{2}$ is an irrational number.

5. (a) 4 (b) 5 (c) 7 (d) 1 (e) 12

6. (a) 1.4142... (b) .333... or $\frac{1}{3}$ (c) 3.1415...
(d) 6.1616... (e) 1.783 (f) no least upper bound

7. (a) 3 (b) 2 (c) 5 (d) $\frac{3}{2}$ (e) 6 (f) $\frac{1}{11}$
(g) $\frac{9}{7}$ (h) 0

8. (a) $2 \cdot 3 = 6$ (b) $4 \cdot 2 = 8$ (c) $9 \cdot 6 = 54$
(d) $11 \cdot 8 = 88$ (e) $0 \cdot 4 = 0$ (f) $1 \cdot 5 = 5$
(g) $\frac{10}{5} = 2$ (h) $\frac{8}{2} = 4$

9. Consider $\frac{b}{a} \in Q$. Since $a > 0$ and $b > 0$, then $\frac{b}{a} > 0$.
There is a positive integer N such that $N > \frac{b}{a}$. Thus, $Na > b$.

10. Given x is a positive rational number. Let 1 be the other rational number. By the hypotheses, there is a positive integer N such that $N \cdot 1 > x$. Thus $N > x$.

5.13 (Time: 3 days)

The purpose of this section is to indicate that the field properties of $(F, +, \cdot, <)$ will help in understanding the arithmetic necessary to handle the irrational numbers under various operations. The student will also gain some practice to develop the skills in

2) $\sqrt{2}$ is the l.u.b. of a set of rational approximations to an element x in the set: $\{x: x \in R \text{ and } x^2 = 2\}$.

3) By the Completeness Property, $\sqrt{2}$ is a real number.

4) We name ℓ by using the approximations in this set. The more approximations we take, the greater the accuracy by representing $\sqrt{2}$.

The Archimedean Property is stated in two seemingly different forms in this section. The second form,

> If a and b are positive real numbers, there is a positive integer N such that $Na > b$.

is probably less obvious than the first and should initiate more discussion than the first. The geometrical interpretation in terms of the lengths of line segments should be stressed. Point out that we really made use of this property when describing the measuring process. When we lay off a string of segments each congruent to a unit segment, along a given line segment, how do we know that we'll ever reach or pass the endpoint? It is the Archimedean Property which guarantees that we will. In many of the standard proofs involving the concept of limit, we rely on the Archimedean Property to select N in terms of a given ϵ. The student will meet some of these situations at the end of Course III.

5.12 Answers to Exercises

1. $3.022\overline{2}$, 3.1847, $3.1999\overline{9}$, 3.201
2. $\{2, 2.2, 2.23, 2.236, 2.2361, \ldots\} = C$

   If ℓ is lub of C then $\ell = 2.2361\ldots$
3. (d), (e)

working with radical expressions.

Particular emphasis should be given to the "enticing conjectures" on page 259 and any others which the students or teacher might bring up for discussion.

The text does not mention the process of rationalizing radical denominators of fractions. The teacher should point out that this process is accomplished with the identity property for $(\cdot)$ of $(F, +. \cdot, <)$. i.e. $\frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}} \cdot 1 = \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}}{2}$.

Following these kinds of problems, the concept of conjugates should be discussed relative to rationalizing radical denominators of fractions. The general property: $\forall a, b \quad (a + b)(a - b) = a^2 - b^2$ should be reviewed. The following type of problems could then be discussed.

$$(3 + \sqrt{2})(3 - \sqrt{2}) = (9 - 2) = 7$$
$$(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5}) = (2 - 5) = (-3)$$
$$(3\sqrt{5} - 2\sqrt{6})(3\sqrt{5} + 2\sqrt{6}) = (45 - 24) = (21)$$

This above concept can then be used in rationalizing radical denominators as in the following simplifications.

$$\frac{1}{2 + \sqrt{3}} = \frac{1}{2 + \sqrt{3}} \cdot \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 2 - \sqrt{3}$$

$$\frac{10}{2\sqrt{3} + 5} = \frac{10}{2\sqrt{3} + 5} \cdot \frac{(2\sqrt{3} - 5)}{(2\sqrt{3} - 5)} = \frac{20\sqrt{3} - 50}{12 - 25} = \frac{20\sqrt{3}-50}{-13}$$

The following exercises will give additional practice in working with radical expression. The teacher should decide whether this additional practice is needed or not.

Simplify each of the following radical expressions. The domain of each variable is the set of positive real numbers.

		(Answers)
1)	$\sqrt{x^4y^5}$	$x^2y^2\sqrt{y}$
2)	$\sqrt{.9x^2}$	$x\sqrt{.9} = \frac{3x\sqrt{10}}{10}$
3)	$\sqrt{\frac{2}{9}}$	$\frac{1}{3}\sqrt{2}$
4)	$(-4\sqrt{42})(-2\sqrt{63})$	$168\sqrt{6}$
5)	$\sqrt{\frac{5}{13}} + \sqrt{\frac{15}{52}}$	$\frac{2}{3}\sqrt{3}$
6)	$\sqrt{9} \cdot \sqrt{3}$	$3\sqrt{3}$
7)	$\sqrt{6x^3y^5} \cdot \sqrt{10x^7y^7}$	$2x^5y^6\sqrt{15}$
8)	$\frac{2}{\sqrt{2} + \sqrt{3}}$	$2\sqrt{3} - 2\sqrt{2}$
9)	$-3\sqrt{5}(2\sqrt{15} - 4\sqrt{5})$	$-30\sqrt{3} + 60$
10)	$(2\sqrt{3} + 5)(3\sqrt{3} - 8)$	$-\sqrt{3} - 22$
11)	$(3 + \sqrt{5})(3 - \sqrt{5})$	4
12)	$\frac{10}{3 + \sqrt{2}}$	$\frac{30 - 10\sqrt{2}}{7}$
13)	$\frac{8}{\sqrt{7} - \sqrt{6}}$	$8(\sqrt{7} + \sqrt{6})$
14)	$\frac{\sqrt{15} - \sqrt{5}}{\sqrt{15} + \sqrt{5}}$	$2 - \sqrt{3}$
15)	$\frac{\sqrt{5} + \sqrt{7}}{2\sqrt{5} + \sqrt{7}}$	$\frac{3 + \sqrt{35}}{13}$

5.14 Answers to Exercises

1. (a) $3\sqrt{5}$ (b) $2\sqrt{6}$ (c) $2x\sqrt{3}$ (d) $4\sqrt{6}$ (e) $\frac{10x}{3}$
(f) $(x - 1)$ (g) $2xy\sqrt{7x}$ (h) $2\sqrt{2}$ (i) $4\sqrt{2}$
(j) $\frac{4x\sqrt{x}}{5}$

2. (a) $\sqrt{12}$ (b) $\sqrt{441}$ (c) $\sqrt{18x^2}$ (d) $\sqrt{1300}$

(e) $\sqrt{20x^2y^2}$ (f) $\sqrt{9x^4y}$ (g) $\sqrt{25a^4b^6}$

3. $\left\{\frac{6}{7\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{7}\right\}$ 4. $\left\{\frac{12\sqrt{2}}{5 - 3\sqrt{2}} \text{ or } \frac{72 + 60\sqrt{2}}{7}\right\}$

5. $\{7, -2\}$ 6. $\{\frac{3}{2}, -\frac{3}{2}\}$ 7. $\{2\sqrt{3}, -2\sqrt{3}\}$

8. $\left\{\sqrt{\frac{29}{3}}, -\sqrt{\frac{29}{3}}\right\}$ 9. $\{-5, -4\}$ 10. $\{-9, 3\}$

11. $\left\{\frac{\sqrt{6} + 5}{\sqrt{3}}\right\}$ or $\left\{\frac{3\sqrt{2} + 5\sqrt{3}}{3}\right\}$ 12. $\{4, 8\}$

13. If a and b are real numbers, $a \geq 0$, $b > 0$,

then $\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{a}{b}$

then $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

14. By counter example

Let $a = 16$ and $b = 9$

$$\sqrt{16} - \sqrt{9} = 4 - 3 = 1 \neq \sqrt{16 - 9}$$

15. If a, b are real numbers such that $a, b \geq 0$

then $(\sqrt{a}\sqrt{b})^2 = \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{a} \cdot \sqrt{b} = (\sqrt{a})^2(\sqrt{b})^2 = ab$

then $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$

then $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

5.16 Answers to Review Exercises

1. (a) 7 (b) 12 (c) 5 (d) 18

2. Assume there is a rational number $\frac{p}{q}$ (p, q are integers, $q \neq 0$) such that $\frac{p}{q} = 7\sqrt{2}$. Then $\sqrt{2} = \frac{p}{7q}$. Since $q \neq 0$, $7q \neq 0$. Thus,

$\sqrt{2}$ is rational. Since the assumption leads to a contradiction, the assumption is false. Therefore, $7\sqrt{2}$ is irrational.

3. Suppose there is a rational number x so that $x^2 = 17$. This means we can find integers p and q where $q \neq 0$ and $(\frac{p}{q})^2 = 17$. Squaring, we get $\frac{p^2}{q^2} = 17$ or $p^2 = 17q^2$. However, p^2 and q^2 each contain an even number of factors of 17 in its complete factorization into primes. Thus, $17q^2$ contains an odd number of factors of 17 in its complete factorization. By the Unique Factorization Principle, $p^2 \neq 17q^2$. Since the assumption leads to a contradiction, the assumption is false and we conclude that there is no rational number whose square is 17.

4. The solution set of the equation is empty if the domain of the variable is Q.

5. (a) $\sqrt{8}$ cm. (b) $\sqrt{50}$ cm. (c) 2 cm. (d) $\sqrt{2a^2}$ cm.

6. 8

7. 2, 2.6, 2.64, 2.645

8. (a) None (b) Can't tell (c) 7.138 (d) None

9. (a) x where $x \in Q$; x where $x \in R$
   (b) None exists in either case

10. (a) For each $a \in S$, $a \geq x$.
    (b) 1. x is a lower bound of S.
        2. If b is any lower bound of S, then $x \geq b$.

11. (a) .11111 (b) No (c) .12 (d) $\frac{1}{9}$

12. (a) $\sqrt{13}$ (b) $\sqrt[3]{8} = 2$ (c) $\sqrt[3]{25}$ (d) $\sqrt[5]{32} = 2$

13. (b) and (d)

14. The least upper bound of a finite set of rational numbers is the largest number in the set. Thus, it must be rational.

Suggested Chapter Test

1) Prove that $x^2 = 7$ has an empty solution set in $(Q, +, \cdot)$ by using the Unique Factorization Property

2) Give an upper bound for the set of all rational numbers less than $\sqrt{2}$. Is there a rational least upper bound?

3) Find 4 approximations to $\sqrt{10}$.

4) Find the least upper bound in Q of the following sets and express it in the form of $\frac{a}{b}$ $b \neq 0$.

   a) $\{.41, .4141, .4141\overline{41}, \ldots\}$

   b) $\{.1, .11, .11\overline{1}, \ldots\}$

   c) $\{.07, .077, .0777\}$

5) Find the greatest lower bound in Q for the set $\{x : x \in Q \text{ and } x^2 \leq 16\}$

6) What is the least upper bound of the set $\{x : x \in Q \text{ and } x^2 \leq 4\}$? Is the least upper bound of the set an element of the set? Answer the same 2 questions for the greatest lower bound.

7) Prove that the set of natural numbers has no upper bound.

8) Simplify the following radical expressions.

   (a) $\sqrt{245x^3y^3}$

   (b) $\sqrt{\frac{25}{9}x^{12}y^{14}}$

   (c) $\sqrt{\frac{7}{30}} + \sqrt{\frac{2}{5}}$

(d) $\sqrt{2}(3\sqrt{3} - 5\sqrt{18})$

(e) $(2\sqrt{5} - \sqrt{3})(3\sqrt{5} + \sqrt{3})$

(f) $\frac{4}{3\sqrt{6} - 4\sqrt{3}}$

9) Find the solution set of each of the following equations in $(Q, +, \cdot)$

(a) $x^2 + 2x - 3 = 0$

(b) $7^2 + x^2 = 25^2$

(c) $x^2 + 4 = 9$

10) Find the solution set of each of the following equations in $(R, +, \cdot)$

(a) $(5\sqrt{2})x - 3 = 17$

(b) $6\sqrt{3} + 6x = 18$

(c) $2x - (3\sqrt{3})x = 5\sqrt{3}$

Answers to Suggested Test Questions

1) Proof: Assume there is an $x \in Q$ such that $x^2 = 7$. Let $x = \frac{p}{q}$ where p, q are integers and $q \neq 0$. Then $\frac{p^2}{q^2} = 7$; then $p^2 = 7q^2$; p^2 contains an even number (possibly 0) of factors of 7. q^2 contains an even number (possibly 0) of factors of 7. Then $7q^2$ contains an odd number of factors of 7.

Therefore $p^2 \neq 7q^2$

Therefore the assumption that $x^2 = 7$ has a solution set in $(Q, +, \cdot)$ is false.

2) An upper bound for this set is:

1.42 or 1.5 or 2, etc.

There is no rational least upper bound.

3) 3, 3.1, 3.16, 3.162.

4) (a) $\frac{a}{b} = \frac{41}{99}$

(b) $\frac{a}{b} = \frac{1}{9}$

(c) $\frac{a}{b} = \frac{777}{10,000}$

5) g.l.b. = (-4)

6) 2, yes

-2, yes

7) Proof: Assume that N has an upper bound. Then, by the Completeness Axiom, every non-empty set of R that has an upper bound has a least upper bound in R. Let this least upper bound be ℓ. Then $n \leq \ell$ for all $n \in N$; but $n + 1 \in N$; Therefore $n + 1 \leq \ell$; therefore $n \leq \ell - 1$.
This contradicts that ℓ is the least upper bound. Therefore our assumption that N has an upper bound is false. Hence N has no upper bound.

8) (a) $7xy\sqrt{5xy}$

(b) $\frac{5x^6y^7}{3}$

(c) $\frac{\sqrt{7}}{2\sqrt{3}}$ or $\frac{\sqrt{21}}{6}$

(d) $3\sqrt{6} - 30$

(e) $-\sqrt{15} - 27$

(f) $\frac{6\sqrt{6} + 8\sqrt{3}}{3}$

9) (a) $\{-3, 1\}$

(b) $\{-24, 24\}$

(c) $\emptyset$

10) (a) $\left\{\frac{4}{\sqrt{2}}\right\}$ or $\{2\sqrt{2}\}$

(b) $\{3, -\sqrt{3}\}$

(c) $\left\{-\left(\frac{45 + 10\sqrt{3}}{23}\right)\right\}$

Chapter 6

Coordinate Geometry

Time Estimate for Chapter 19 - 23 days

The main objective of this chapter is that of uniting the real number system with axiomatic geometry. The realization of this overall goal is accomplished through a series of specific objectives:

1. extend the axiomatic geometry of Chapter 3 by the addition of three new axioms, based primarily upon the students' knowledge of the ruler and the real number system.
2. relate the structure of real numbers to the structure of the line.
3. understand the nature of a line coordinate system and system and related properties including betweeness of points, division points, segments, rays and distances on a line.
4. extend this concept to a plane coordinate system and properties with sets of points, particularly those sets forming lines in the plane.
5. study the equation for a line in a plane with emphasis on substitution, slope, parallel lines and intersecting lines.
6. use numbers, ordered pairs of numbers, equations and inequalities as aids in the investigation of properties of geometric figures.

7. gain an understanding of basic properties of triangles and parallelograms, concurrence of lines and other various affine properties.
8. master the Pythagorean property of right triangles within a plane rectangular coordinate system.
9. Finally, throughout the chapter, develop the ability to formulate definitions and construct proofs based upon the ordered field properties of real numbers.

6.1 Introduction (Time for 6.1, 6.2, 6.3 = 1 day)

This is a brief review of the axioms stated in Chapter 3 and should be assigned for reading at home before class and/or serve as a basis for a series of questions initiating the lesson centered about section 6.2.

6.2 Axiom 4: Uniqueness of Line Coordinate System

Starting with an unmarked line, students should be called upon to draw various rulers on the line; the eventual realization of this activity will be that the naming of any 2 points with coordinates taken from the Real number system will determine the coordinates for all other points on the line. It is usual that the points taken are the O and I points, matched with the real numbers 0 and 1 respectively, forming the O,I-coordinate system. Please note order is imperative: the O,I-coordinate system ≠ the I,O-coordinate system.

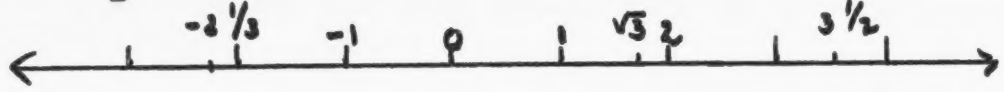
In establishing coordinate systems on a line, students should realize that there are as many points on a line as there are real numbers. Only under this assumption can we set up a one-to-one correspondence.

In Section 6.3, problem No. 2 and parts of No. 3 lend themselves to a development of the lesson. Some care should be taken in clarifying the wording in No. 2b.

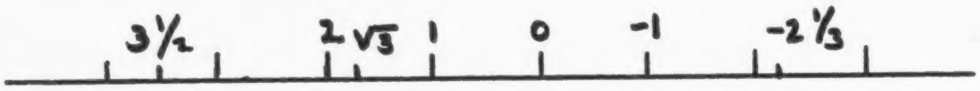
6.3 Exercises

1. O, I, $\frac{1}{2}$

2a.



b.



3. a. 2, -1
   b. -1, 2
   c. 2, $2\frac{1}{2}$
   d. 2, $\frac{2}{3}$
   e. $\frac{1}{6}$, $\frac{2}{3}$

4. a. False
   b. True
   c. True
   d. True
   e. True
   f. False
   g. True
   h. False

5. a. (1) The A, B-coordinate of A and B are 0 and 1 respectively. A, B-distance = $|0 - 1| = 1$.
   (2) The A, B-distance between B and A = $|1 - 0| = 1$
   (3) Let the A, B-coordinates of P and Q be p and q respectively. If P = Q, then p=q and

$|p-q| = 0.$

If $|p-q| = 0$, $p = q$ and $P = Q$.

b. (1) Let the A, B-coordinate of S be s.

$|s-1| = 2|s-0|$

(2) If $0<s<1$ Then $1-s = 2s$ or $s = \frac{1}{3}$.

(3) If $s<0<1$ Then $-s+1 = -2s$ or $s = -1$

(4) The case $0<1<s$ is ruled out because then S would be nearer B than it is to A.

6.4 Axiom 5: Relating Two Coordinate Systems on a Line

(Time: 6.4, 6.5 = 2 days)

The basic motivation of this lesson springs from the standard ruler (the teacher may wish to use a yardstick for demonstration purposes). Two basic coordinate systems are identified in the process of measuring: one of inches and one of feet. Each point is then assigned two coordinates, one from each system.

The purpose of this section becomes apparent - to develop methods of relating two coordinate systems by studying the two coordinate names assigned to each point. From notions of dilations ($x \to ax$ where $a \neq 0$) and translations ($x \to x + b$), the relation from one coordinate system to another is seen as a composite mapping ($x \to ax+b$ where $a \neq 0$).

From $x \to ax+b$ and $x \to x'$, the equation $x' = ax+b$ is developed. In this sense, the yardstick first used demonstrates the equation $i = 12f$ (or $x' = 12x+0$) where $x=f=$ measure in feet and $x' = i =$ measure in inches.

Practice in algebraic manipulations may be necessary beyond the 2 day time estimation and, if this is the case, the teacher should provide review problems while continuing to teach the material in sections to follow.

6.5 Exercises

1. a. 5 e. 8999 h. $\frac{\sqrt{11} + 1}{3}$
b. 11 f. $1000\frac{1}{3}$ i. $\frac{1}{2}$
c. 3 g. $3\sqrt{10} - 1$ j. 1
d. -7

2. a. 4 e. 1503 h. $2(\sqrt{11} - 3)$
b. 5 f. 5994 i. 6
c. 10 g. $\frac{1}{2}\sqrt{10} + 3$ j. 2
d. -50

3. a. $x' = x+3$ f. $x' = -x - 1$
b. $x' = x-9$ g. $x' = \frac{1}{2}x - 1$
c. $x' = -x$ h. $x' = -2x + 4$
d. $x' = 2x$ i. $x' = 2x + 200$
e. $x' = 3x$ j. $x' = 2x - 36$

4. a. $y = -\frac{1}{3}x + \frac{2}{3}$
b. $x = -3y + 2$

5. a. $x = -2y + 2$ d. $z = 2x + 2$
b. $x = \frac{1}{2}z - 1$ e. $y = -\frac{1}{2}x + 1$
c. $y = -\frac{1}{4}z + \frac{3}{2}$ f. $z = -4y + 6$

6. a. $F = \frac{9}{5}C + 32$ e. $(F = \frac{9}{5}F + 32) \Rightarrow (F = -40)$

b. $C = \frac{5}{9}F - \frac{160}{9}$ f. $(C+20 = \frac{9}{5}C+32) \Rightarrow (C = -15) \Rightarrow (F = 5)$

c. 122; -4, 1832 g. Affine mapping

d. 10; $-25\frac{5}{9}$; $1093\frac{1}{3}$.

7. Let x = A, B-coordinate of a point on $\overleftrightarrow{AB}$ and y its B, A-coordinate. Then x-coordinates of A and B are 0 and 1 respectively, while y-coordinates of A and B are 1 and 0 respectively.
If $y = ax + b$, $1 = a \cdot 0 + b$ and $0 = a \cdot 1 + b$.
Therefore $b = 1$ and $a = -1$, and $y = -x + 1$. To find the point where A, B-coordinate is equal to its B, A-coordinate let $y = x$. This leads to $x = y = \frac{1}{2}$.

8. x' - distance $PQ = |-5 - 3| = 8$
x - distance $PQ = |4 - 2| = 2$
The ratio of distances $\frac{x'}{x} = \frac{8}{2} = \frac{4}{1}$.
$x' = -4x + 11$.
$a = -4$; $|-4| = \frac{4}{1}$

9. Since $x' = ax + b$ connects A, B-coordinates to A', B' - coordinates $p' = ap + b$ and $q' = aq + b$.
The A', B' - distance $PQ = |p' - q'|$
$$= |(ap + b) - (aq + b)| = |ap - aq| = |a||p - q|.$$
The A, B-distance $PQ = |p - q|$.
Therefore the A', B'-distance $PQ = |a|$, the A, B-distance PQ.

6.6 Segments, Rays, Midpoints (Time: 6.6, 6.7 = 2 - $2\frac{1}{2}$ days)

Although this section recalls basic geometric terms previously studied, its main purpose is that of redefining these terms in the light of coordinate systems.

A critical definition is given for the property of betweeness, a term known by students solely on an intuitive basis. From "betweeness" springs a discussion and definition of a line segment, endpoints, interior points and midpoints; all of these terms are viewed as inequalities or specific values on a line coordinate system. Extension to the definition of a ray follows naturally.

Selected items from problems 1, 3 and 6 in Section 6.7 should be done in class before assigning homework. Students might find the proofs in problem 10 difficult and challenging but, once two or three are demonstrated in class, most of these proofs should be within the grasp of many of the students. Although these proofs may be omitted, the proofs in problems 7, 8 and 9 should be done.

6.7 Exercises

1. a. $-1 \leq x \leq 1$
   b. $-1 \leq x \leq 2$
   c. $1 \leq x \leq 2$
   d. $x \geq -1$
   e. $x \leq 1$
   f. $x \leq 1$
   g. $x = 0$
   h. $x = \frac{1}{2}$
   i. $0 \leq x \leq 1$
   j. $-1 \leq x \leq 2$

2. a. $5\frac{1}{2}$
   b. $2\frac{1}{2}$
   e. 0
   f. $\frac{5}{12}$
   i. $\frac{\sqrt{2} + \sqrt{3}}{2}$

c. $-5\frac{1}{2}$ g. $\frac{7}{20}$

d. $-2\frac{1}{2}$ h. 2.3

3. a. 13 b. -19 c. -3 d. $2\sqrt{2} - 3$

4. a. B is between A and C. c. L is between M and N.

b. R is between P and Q. d. E is between D and F.

5. $z = \frac{1}{2}(3x - 2) + 1 = \frac{3}{2}x.$

6. a. $\overrightarrow{BC}$ or $\overrightarrow{BD}$, etc.

b. $\overrightarrow{FE}$ or $\overrightarrow{FD}$, etc.

c. F

d. $\overline{CE}$

e. $\overrightarrow{AE}$

f. open half line DF (or $\overrightarrow{DF} - D$).

g. $\overline{CE} - C$ (or $\overline{CE}$ where C is not included).

h. $\overline{AB} - A - B$ (or $\overline{AB}$ where points A and B are not included).

7. $\underline{X}$ is in $\overline{AB}$. Therefore the A, B-coordinate x of $\underline{X}$ satisfies the condition $0 < x < 1$. Thus $x > 0$ and is therefore in $\overrightarrow{AB}$.

8. Let the A, B-coordinate of $\underline{X}$ be x and the A, B-coordinate of $\underline{Y}$ be y.

(1) Then $0 < x < 1$ and $0 < y < 1$.

(2) Since $\underline{X} \neq \underline{Y}$, $x \neq y$.

(3) Either $y > x$ or $y < x$.

(4) Let $y > x$. Then $0 < x < y < 1$.

(5) If Z is a point in $\overline{XY}$, then $x < z < y$.

(6) Therefore $0 < z < 1$ and Z is in $\overline{AB}$.

(Similar case for $y < x$).

9. Theorem 2: If C is between P and Q, then $PC + CQ = PQ$

Proof: To show $PC + CQ = PQ$, demonstrate $|p - c| + |c - q| = |p - q|$. $|p - c| + |c - q| = (p - c) + (c - q) = (p + -c) + (c + -q) = p + (-c + c) + -q = p + 0 + -q = p + -q = p - q = |p - q|$.

*10. a. Q = midpoint of $\overline{PR}$. Solving $\frac{q - p}{r - p} = \frac{1}{2}$ for q, we see $q = \frac{1}{2}(p + r)$.

b. $\frac{q - p}{r - p} = \frac{1}{3}$ implies Q = trisection point of $\overline{PR}$ nearer P;
$\frac{q - p}{r - p} = \frac{2}{3}$ implies Q = trisection point of $\overline{PR}$ nearer R.

c. Since Q is an interior point of $\overline{PR}$, we know that $\frac{q - p}{r - p}$ must be a fractional value less than 1.

Intuition tells us that $m = 0$ and $n = 1$.

Proof: Given $m < \frac{q - p}{r - p} < n$ and Q between P and R, then $m = 0$ and $n = 1$.

(1) Let $m = 0$ and $n = 1$. Either $p > r$ or $p < r$.

(2) $p > r \Rightarrow r - p$ is negative $\Rightarrow (0 < \frac{q - p}{r - p} < 1) \Rightarrow (0 > q - p > r - p) \Rightarrow (p > q > r) \Rightarrow$ Q is between P and R.

(3) $p < r \Rightarrow (0 < \frac{q - p}{r - p} < 1) \Rightarrow (0 < q - p < r - p) \Rightarrow (p < q < r) \Rightarrow$ Q is between P and R.

Alternate Proof:

(1) Either $(p < q < r)$ or $(p > q > r)$.

(2) $(p < q < r) \Rightarrow (0 < q - p < r - p) \Rightarrow (0 < \frac{q - p}{r - p} < 1) \Rightarrow m = 0$ and $n = 1$.

(3) $(p > q > r) \Rightarrow (0 > q - p > r - p) \Rightarrow (0 < \frac{q - p}{r - p} < 1) \Rightarrow m = 0$ and $n = 1$.

Check: Let $p = -2$ and $r = 8$. Then $(0 < \frac{q - p}{r - p} < 1) \Rightarrow$ $(0 < \frac{q - (-2)}{8 - (-2)} < 1) \Rightarrow (0 < \frac{q + 2}{10} < 1) \Rightarrow$ $(0 < q + 2 < 10) \Rightarrow (-2 < q < 8) \Rightarrow$ Q is an internal point of $\overline{PR}$.

d. From part c, $p = -2$ and $r = 8$. Select $q < -2$, say $q = -4$. Then $\frac{q - p}{r - p} = \frac{-4 - (-2)}{8 - (-2)} = \frac{-2}{10}$ = a NEGATIVE value.

Proof: If P is between Q and R, then $\frac{q - p}{r - p} < 0$.

(1) Either $q < p < r$ or $q > p > r$.

(2) $(q < p < r) \Rightarrow (q < p$ and $p < r) \Rightarrow (q - p < 0$ and $0 < r - p) \Rightarrow \frac{q - p}{r - p} < 0$.

(3) $(q > p > r) \Rightarrow (q > p$ and $p > r) \Rightarrow (q - p > 0$ and $0 < r - p) \Rightarrow \frac{q - p}{r - p} < 0$.

e. Proof: If $\frac{q - p}{r - p} > 1$, then R is between P and Q.

(1) Either $p > r$ or $p < r$.

(2) $(p > r) \Rightarrow (0 > r - p)$.
$(\frac{q - p}{r - p} > 1) \Rightarrow (q - p < r - p) \Rightarrow (q < r)$.
$(p > r)$ and $(q < r) \Rightarrow (p > r > q)$.

(3) $(p < r) \Rightarrow (0 < r - p)$.
$(\frac{q - p}{r - p} > 1) \Rightarrow (q - p > r - p) \Rightarrow (q > r)$.
$(p < r)$ and $(q > r) \Rightarrow (p < r < q)$.

f. Given $\frac{q - p}{r - p}$

Let $q' = aq + b$; $p' = ap + b$; $r' = ar + b$

Then $\frac{q' - p'}{r' - p'} = \frac{(aq + b) - (ap + b)}{(ar + b) - (ap + b)} = \frac{aq - ap}{ar - ap} = \frac{q - p}{r - p}$

g. To show $\frac{q - p}{r - p}$ is the P, R-coordinate of Q, let $p = 0$ and $r = 1$.

Then $\frac{q - p}{r - p} = \frac{q - 0}{1 - 0} = \frac{q}{1} = q =$ P, R-coordinate of Q.

Alternate Proof: Find the formula $x' = ax + b$ that converts p to 0 and r to 1. This is $x' = \frac{x - p}{r - p}$ as can be checked with $x = p$ and $x = r$. Then when $x = q$, we get the P, R-coordinate of Q to be $\frac{q - p}{r - p}$.

h. Proof (I): $\frac{q - p}{r - p} = 0$ if and only if Q = P.

(1) $(\frac{q - p}{r - p} = 0) \Rightarrow (q - p = 0) \Rightarrow (q = p) \Rightarrow Q = P$.

(2) $(Q = P) \Rightarrow (q = p) \Rightarrow (q - p = 0)$
Since $P \neq R$, $p \neq r$ and $r - p \neq 0$
$(q - p = 0)$ and $(r - p \neq 0) \Rightarrow (\frac{q - p}{r - p} = 0)$.

Proof (II): $\frac{q - p}{r - p} = 1$ if and only if Q = R.

(1) $(\frac{q - p}{r - p} = 1) \Rightarrow (q - p = r - p) \Rightarrow (q = r) \Rightarrow$
$Q = R$.

(2) $(Q = R) \Rightarrow (q = r)$
Since $P \neq R$, $p \neq r$ and $r - p \neq 0$
$(q = r) \Rightarrow (q - p = r - p) \Rightarrow (\frac{q - p}{r - p} = 1)$.

6.8 Axiom 6: Parallel Projections and Line Projections

(Time: 6.8, 6.9 = 1 - $1\frac{1}{2}$ days)

Although the wording of this last axiom might prove to be cumbersome and confusing to some students, the concept is reasonably simple. Essentially, one line may be transformed to a second line by means of a parallel projection; Axiom 6 states that parallel projections preserve coordinate systems.

The six axioms are restated at the end of this section to aid the students in proving statements found in section 6.9. Certainly, at least two of these proofs should be demonstrated in class before assigning the remaining exercises for homework.

6.9 Exercises

1. Given B is between A and C.
   By Axiom 4, let A and C correspond to 0 and 1 respectively.
   Then A, C-coordinate of B is between 0 and 1.
   By Axiom 5, let $x' = ax + b$ where $a = 1$ and $b = 0 \Rightarrow x' = x$.
   By Axiom 6, the A', C'-coordinate of B' is between 0 and 1.
   Therefore, B' is between A' and C'.

2. Given B is the midpoint of $\overline{AC}$.
   The A, C-coordinate of B $= \frac{1}{2}$.
   By Axiom 6, the A', C'-coordinate of B' $= \frac{1}{2}$
   Therefore, B' is the midpoint of $\overline{A'C'}$.

3. By definition, all A, C-coordinates x of $\overrightarrow{AC}$ satisfy $x \geq 0$.
   By Axiom 6, the A', C' - coordinates of images of points in $\overleftrightarrow{AC}$ = their A, C-coordinates. Thus $x' = x$.
   $x' \geq 0$ implies $\overrightarrow{A'C'}$ is a ray.

4. Given B divides $\overline{AC}$, from A to C, in the ratio r.
   Let A, C-coordinate of B $= \frac{b - a}{c - a} = b$.
   Then $x' = \frac{x - a}{c - a}$
   By Axiom 6, the A', C'-coordinate of B' $= \frac{b - a}{c - a} = b$.
   Therefore, B' divides $\overline{A'C'}$, from A to C. in the ratio r.

5. To prove $\dfrac{|a - c|}{|c - b|} = \dfrac{|a' - c'|}{|c' - b'|}$, use Axiom 6 with the A, B-coordinates and A', B'-coordinates equal.

   Then $\dfrac{|0 - c|}{|c - 1|} = \dfrac{|0 - c'|}{|c' - 1|}$ But A', B'-coordinate c = c.

   Therefore, $\dfrac{|0 - c|}{|c - 1|} = \dfrac{|0 - c|}{|c - 1|}$

6. Consider the A, B-coordinate system where A = 0, B = 1 and D = $\frac{1}{2}$ and the parallel projection from $\overleftrightarrow{AB}$ to $\overleftrightarrow{AC}$ in the direction of $\overleftrightarrow{BC}$.

   By Axiom 6, the A, C-coordinate system (A = 0 and C = 1) must take the image of D to $\frac{1}{2}$

   Therefore, the line containing D and parallel to $\overleftrightarrow{BC}$ passes through the midpoint of $\overline{AC}$.

7. Under the A, B-coordinate system A = 0, B = 1 and D = $\frac{1}{3}$ and the parallel projection maps $\overleftrightarrow{AB}$ to $\overleftrightarrow{AC}$ in the direction of $\overleftrightarrow{BC}$.

   By Axiom 6, the A, C-coordinate system (A = 0 and C = 1) must take the image of D to $\frac{1}{3}$

   Therefore, the line containing D and parallel to $\overleftrightarrow{BC}$ trisects $\overleftrightarrow{AC}$ nearer to A.

6.10 Plane Coordinate Systems (Time: 6.10, 6.11 = $1\frac{1}{2}$ - 2 days)

The purpose of this section is to extend, in a natural way, what has been learned about line coordinate systems to the existence and study of plane coordinate systems.

A plane coordinate system is determined by the intersection of two line coordinate systems. By calling the O, I-line coordinate system the x-axis and the O, J-line coordinate system the y-axis, there is now formed the O, I, J-plane coordinate system. Again, order is imperative: O, I, J-coordinate system $\neq$ O, J, I - coordinate system.

It should be noted that the distance from O to I need not equal the distance from O to J and that the x and y axes need not be perpendicular.

The teacher should have various students graphing the same sets on the board at the same time to show a variety of plane coordinate systems whose conditions, once graphed, appear to be different but are equivalent. (See Section 6.11, No. 9 for examples).

6.11 Exercises

1. All points on the y-axis have 0 as their x-coordinate, and the y-coordinate can be any real number.

2. All these points are on the x-axis and only those whose x-coordinates are negative. The set of points forms an open half-line.

3. a. $\{P(x,y) \mid x = 0\}$. d. $\{P(x,y) \mid x < 0 \text{ and } y > 0\}$.
b. $\{P(x,y) \mid x = 0, y > 0\}$ e. $\{P(x,y) \mid x<0 \text{ and } y<0\}$
c. $\{P(x,y) \mid x = 0 \text{ and } y < 0\}$ f. $\{P(x,y) \mid x>0 \text{ and } y<0\}$.

4. a. For any point on ℓ, there is only one line containing that point and parallel to $\overleftrightarrow{OJ}$, and that line is ℓ itself. Therefore all points of ℓ have 1 as x-coordinate. Conversely, if a point has 1 as

5. To prove $\frac{|a - c|}{|c - b|} = \frac{|a' - c'|}{|c' - b'|}$, use Axiom 6 with the A, B-coordinates and A', B'-coordinates equal.

   Then $\frac{|0 - c|}{|c - 1|} = \frac{|0 - c'|}{|c' - 1|}$ But A', B'-coordinate c = c.

   Therefore, $\frac{|0 - c|}{|c - 1|} = \frac{|0 - c|}{|c - 1|}$

6. Consider the A, B-coordinate system where A = 0, B = 1 and D = $\frac{1}{2}$ and the parallel projection from $\overleftrightarrow{AB}$ to $\overleftrightarrow{AC}$ in the direction of $\overleftrightarrow{BC}$.

   By Axiom 6, the A, C-coordinate system (A = 0 and C = 1) must take the image of D to $\frac{1}{2}$

   Therefore, the line containing D and parallel to $\overleftrightarrow{BC}$ passes through the midpoint of $\overline{AC}$.

7. Under the A, B-coordinate system A = 0, B = 1 and D = $\frac{1}{3}$ and the parallel projection maps $\overleftrightarrow{AB}$ to $\overleftrightarrow{AC}$ in the direction of $\overleftrightarrow{BC}$.

   By Axiom 6, the A, C-coordinate system (A = 0 and C = 1) must take the image of D to $\frac{1}{3}$

   Therefore, the line containing D and parallel to $\overleftrightarrow{BC}$ trisects $\overleftrightarrow{AC}$ nearer to A.

6.10 Plane Coordinate Systems (Time: 6.10, 6.11 = $1\frac{1}{2}$ - 2 days)

The purpose of this section is to extend, in a natural way, what has been learned about line coordinate systems to the existence and study of plane coordinate systems.

A plane coordinate system is determined by the intersection of two line coordinate systems. By calling the 0, I-line coordinate system the x-axis and the 0, J-line coordinate system the y-axis, there is now formed the 0, I, J-plane coordinate system. Again, order is imperative: 0, I, J-coordinate system $\neq$ 0, J, I - coordinate system.

It should be noted that the distance from 0 to I need not equal the distance from 0 to J and that the x and y axes need not be perpendicular.

The teacher should have various students graphing the same sets on the board at the same time to show a variety of plane coordinate systems whose conditions, once graphed, appear to be different but are equivalent. (See Section 6.11, No. 9 for examples).

6.11 Exercises

1. All points on the y-axis have 0 as their x-coordinate, and the y-coordinate can be any real number.

2. All these points are on the x-axis and only those whose x-coordinates are negative. The set of points forms an open half-line.

3. a. $\{P(x,y) \mid x = 0\}$. d. $\{P(x,y) \mid x < 0 \text{ and } y > 0\}$.
   b. $\{P(x,y) \mid x = 0, y > 0\}$ e. $\{P(x,y) \mid x<0 \text{ and } y<0\}$
   c. $\{P(x,y) \mid x = 0 \text{ and } y < 0\}$ f. $\{P(x,y) \mid x>0 \text{ and } y<0\}$.

4. a. For any point on ℓ, there is only one line containing that point and parallel to $\overleftrightarrow{OJ}$, and that line is ℓ itself. Therefore all points of ℓ have 1 as x-coordinate. Conversely, if a point has 1 as

x-coordinate, it must be on ℓ; otherwise its x-coordinate is not 1. $\ell = \{P(x,y) \mid x = 1\}$.

b. $m = \{P(x,y) \mid y = 1\}$.

c. $\ell \cap m = \{P(x,y) \mid x = 1 \text{ and } y = 1\}$.

5. There is exactly one point in $\overleftrightarrow{OI}$ whose O, I-coordinate is a, call it A. There is exactly one line through A parallel to $\overleftrightarrow{OJ}$, call it ℓ. Similarly, there is exactly one point in $\overleftrightarrow{OJ}$ whose O, J-coordinate is b, call it B. And there is exactly one line through B parallel to $\overleftrightarrow{OI}$, call it m. $\ell \cap m$ contains exactly one point, and this is the point whose coordinates are (a, b).

6. a. $\{P(x,y) \mid y = 4\}$

   b. $\{P(x,y) \mid x = 3\}$

   c. P has the coordinates (3, 4). Let B have coordinates (3, 5) and C have coordinates (4, 4). Then the set $\{Q(x,y) \mid y > 4 \text{ and } x > 4\}$ is the interior of $\angle BPC$.

   d. Using the points P, C and D(3, 3)
   $\{R(x,y) \mid y \leq 4 \text{ and } x > 0\} = \angle DPC - \overrightarrow{PD}$

7. a. $\{P(x,y) | y = 2\}$

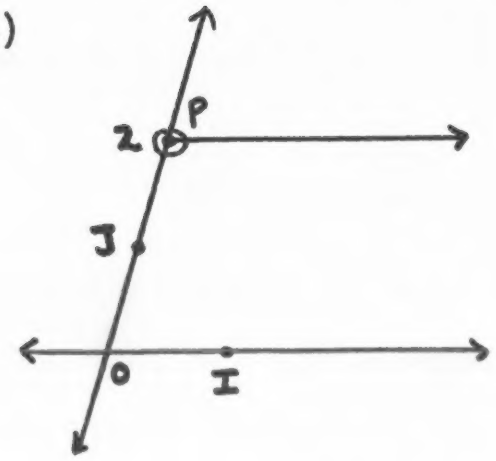
   b. $\{P(x,y) | x = -3\}$

8. a. $\{P(x,y) | y = -5\}$

   b. $\{P(x,y) | x = -4\}$.

9.

a)



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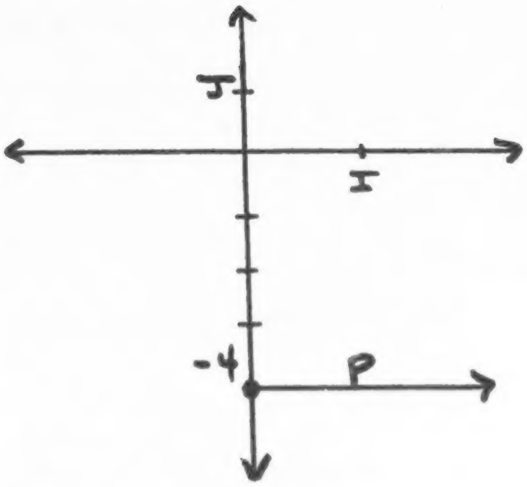

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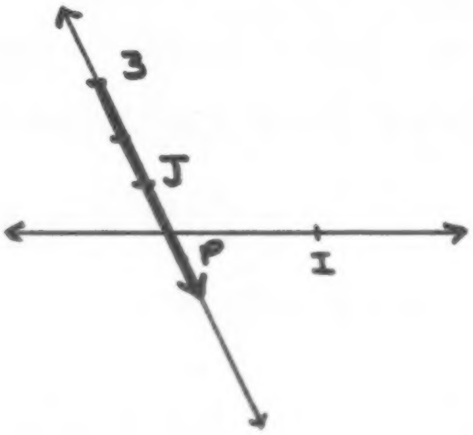

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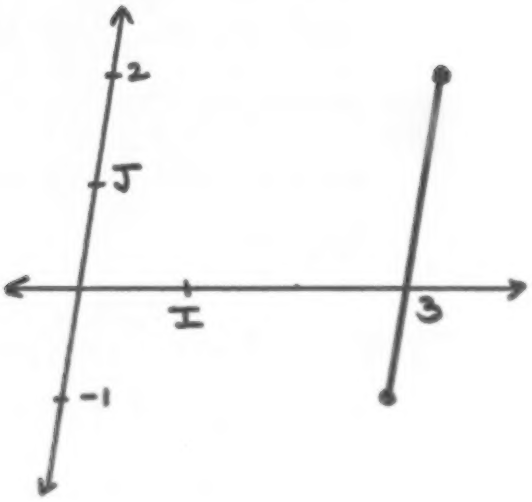
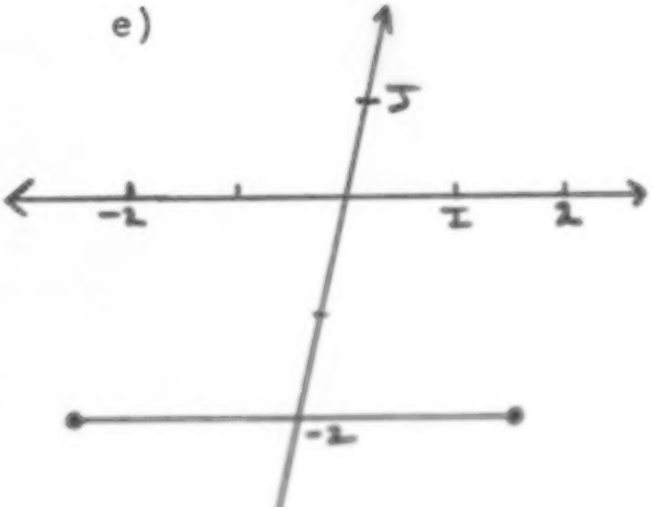


e)



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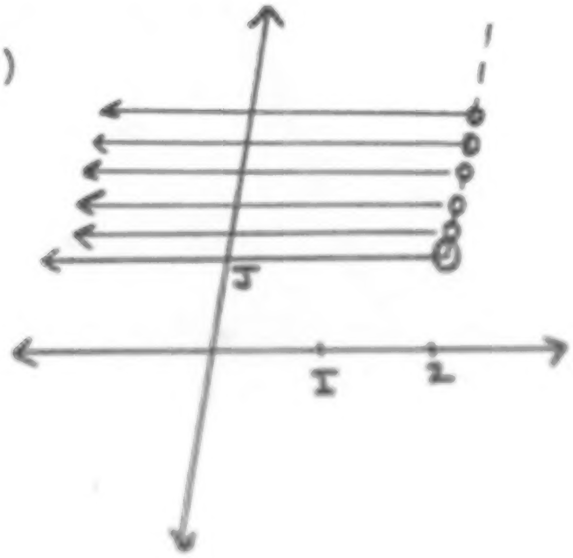
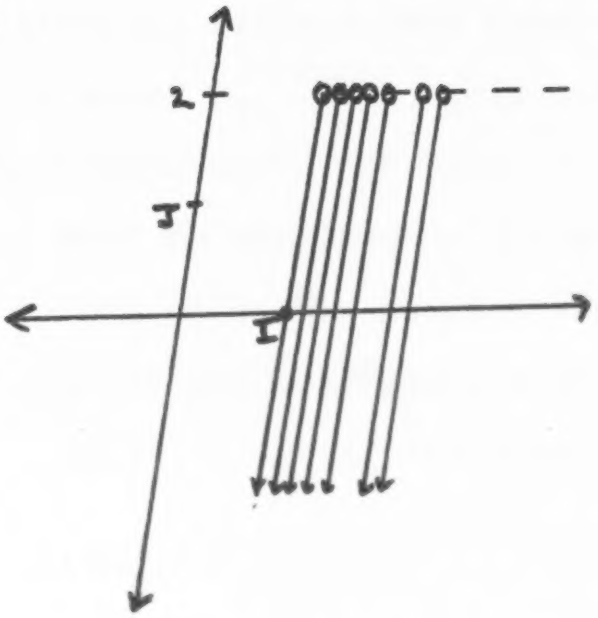
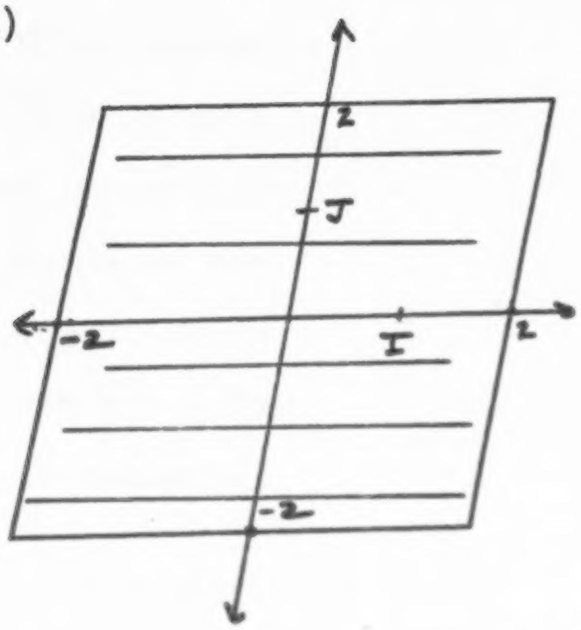


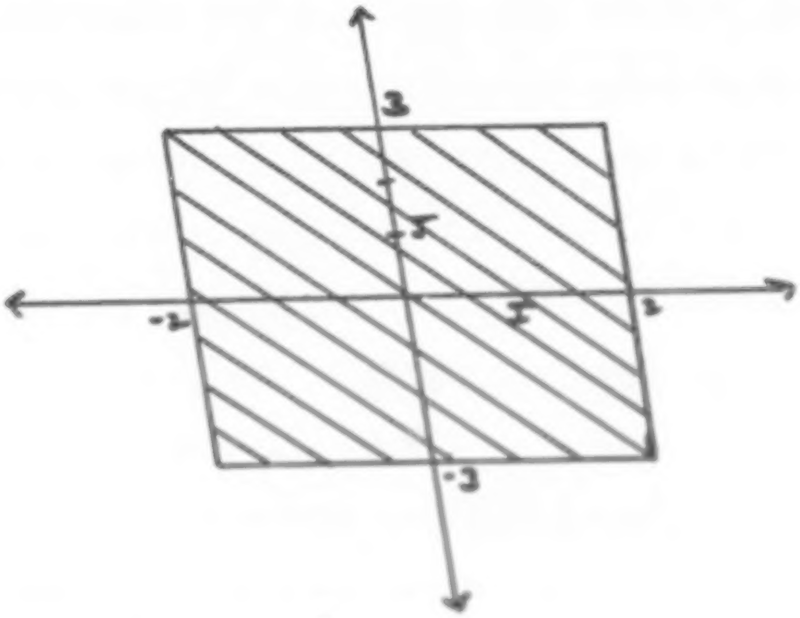
g)



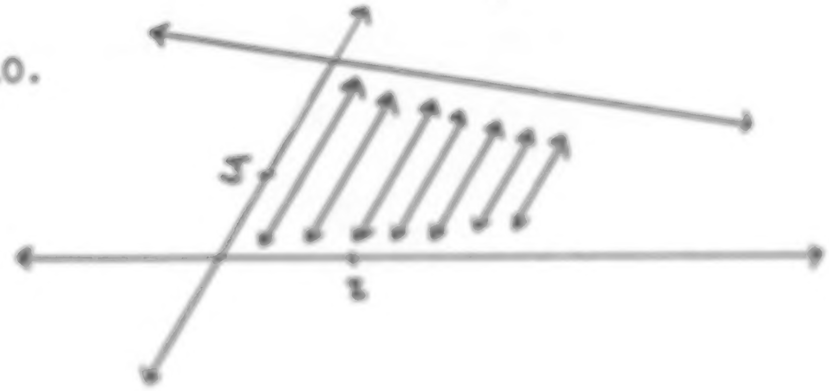
h)



i)



10.



The x-coordinates of points on ℓ are the O, I-coordinates of points on $\overleftrightarrow{OI}$ acquired under the parallel projection f from ℓ to $\overleftrightarrow{OI}$ in the direction of $\overleftrightarrow{OJ}$. By Axiom 6, if $O \xrightarrow{f} O'$ and $I \xrightarrow{f} I'$ then the x-coordinate of any point on ℓ is the O', I'-coordinate of that point. Hence the correspondence between x-coordinates of points of ℓ and their points is a coordinate system-namely the coordinate system with base (O', I').

6.12 An Equation for a Line (Time: 6.12, 6.13 = 2 days)

The purpose of this section is that of gaining skill in describing sets of points in the plane that form straight lines by means of equations. Since students have had experience with equations for lines that were parallel to one of the axes, the main concern here will be with equations of the form $ax + by + c = 0$ where $a \neq 0$ and $b \neq 0$ (lines that intersect both axes).

The main approach taken is that of substitution. In section 6.15, the concept of slope and the slope-point form of the equation are introduced via exercises, and the teacher is referred to the Teachers Commentary for section 6.14.

Since the claim is made that every line in the plane can be described by the equation $ax + by + c = 0$, exercise no. 2 in section 6.13 is important. Exercises nos. 6, 7, 8 and 9 are essential; selection may be made from the remaining exercises in section 6.13.

6.13 Exercises

1. a. A is a point of ℓ b. B is c. C is not
d. D is e. E is not f. F is

g. G is
h. H is not
i. K is not
j. L is
k. M is
l. N is

2. a, d, e and f can be equations for lines.

3. a. $y = -3x + 5$
   b. $y = -3x + 8$
   c. not possible
   d. $y = \frac{2}{5}$
   e. $y = -2x + 8$
   f. $y = -\frac{a}{b}x + \frac{c}{b}$

4. There are an infinite number of correct answers for each. We select one pair arbitrarily for each.
   a. $P(0, 2)$ is on the line $Q(1, 2)$ is not.
   b. $P(6, 0)$ is on the line $Q(6, 1)$ is not.
   c. $P(4, 0)$ is on the line $Q(4, 1)$ is not.
   d. $P(9\frac{1}{3}, 8)$ is on the line $Q(9, 8)$ is not.
   e. $P(8, 0)$ is on the line $Q(8, 1)$ is not.
   f. $P(12, \frac{\sqrt{3}}{5})$ is on the line $Q(12, 2)$ is not.

5. a. Does contain.
   b. Does
   c. Does
   d. Does not
   e. Does not
   f. Does
   g. Does not.
   h. Does
   i. Does
   j. Does not
   k. Does not
   l. Does not

6. Note that the coordinates of D are (0, 0), of I are (1, 0), of J are (0, 1).
   a. 0
   b. J
   c. I
   d. 0
   e. I
   f. J
   g. 0
   h. 0, J

7. Any equation equivalent to those listed for each part is acceptable.
   a. $y = 0$
   b. $x - y - 2 = 0$
   c. $x + y = 6$
   d. $x + y = 0$
   e. $x - 4y + 2 = 0$
   f. $y = -2x + 22$
   g. $y = \frac{1}{2}x - 3$
   h. $3y = 5x + 15$

8. a. $y = 0$ b. $x = 0$ c. $x + y = 1$

9. a. $\overline{OI} = \{P(x,y) \mid y = 0 \text{ and } 0 \le x \le 1\}$
   b. $\overline{OJ} = \{P(x,y) \mid x = 0 \text{ and } 0 \le y \le 1\}$
   c. $\overline{IJ} = \{P(x,y) \mid x + y = 1,\ 0 \le x \le 1\}$

10. $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2})$.

11. a. $(2, 1)$
   b. $(4, 2)$
   c. $(-1, 4)$
   d. $(2, -2\frac{1}{2})$
   e. $(0, 0)$
   f. $(\frac{9}{4}, -\frac{7}{4})$

6.14 Intersection of Lines (Time: 6.14, 6.15 = $2\frac{1}{2}$ to 3 days)

The purpose of this section is to solve systems involving pairs of linear equations. Cases involving the intersection of one point and the null intersection are studied. The teacher may wish to mention the situation where the equations are dependent ($2x + y = 5$ and $4x + 2y = 10$) thus resulting in an intersection that includes an infinite set of points ($\{(x,y) \mid x = k \text{ and } y = 5 - 2k,\ k \in \text{Real numbers}\}$).

In section 6.15, problems 5, 6, 7, 8 introduce and develop the concept of the slope of a line and the point-slope form of an equation. The teacher should feel free to develop this material in class or to provide additional lessons concerning slope. Experience has shown that teacher explanation and class time are necessary in this area.

In studying parallel lines given in the form of equations ($\underline{y = ax + b}$ and $\underline{y = ax + c}$), the teacher should attempt to bring the following notions into a classroom discussion:

1. If $b \neq c$, the lines will be parallel such that their intersection is empty.
2. If $b = c$, the lines will be dependent. Since both equations are equal, only one line is described. The intersection will be the line itself. Parallelism must still hold to maintain an equivalence relation; certainly parallelism is reflexive and every line must be parallel to itself.
3. Equations of the form $y = ax + b$ can be readily derived from equations in the point-slope form.

   Ex: Let ℓ contain points (3, 5) and (4, 7) and let any point on the line ℓ be given by the general coordinates (x, y).

   $$\frac{y - 5}{x - 3} = \frac{7 - 5}{4 - 3}$$

   $$\frac{y - 5}{x - 3} = \frac{2}{1} \qquad \text{NOTE: slope} = \frac{2}{1} = 2$$

   $$y - 5 = 2(x - 3)$$

   $$y - 5 = 2x - 6$$

   $$y = 2x - 1$$

   Comparing $y = 2x - 1$ to the general form $y = ax + b$, we see that "a" is the slope of the equation.

4. An investigation of "b" as the y-intercept in the equation $y = ax + b$ is optional. Students may discover this and bring it to the attention of the teacher and the class. However, a grasp of the point-slope form seems to be of greater conceptual value than the slope-intercept method.

5. The parallel postulate should be investigated and students should be given problems of the type, "Given line ℓ and point P, find the equation of the line parallel to ℓ and containing P."

   Ex: Let $\ell = \{(x,y) \mid y = 3x - 2\}$. Find the equation for line k where $k \parallel \ell$ and point $(3, 5) \in k$. Since $y = 3x - 2$ has a slope of 3 and $k \parallel \ell$, we get $(y = ax + b) \Rightarrow (y = 3x + b)$ for line k. By substituting, $(y = 3x + b) \Rightarrow (5 = 3 \cdot 3 + b)$ $\Rightarrow (-4 = b)$.

   $\therefore$ k has an equation $y = 3x - 4$.

   NOTE: (See Section 3.9, problem 3)

6. The same situation should be studied by the point-slope form: Taking the general point (x, y), point P (3, 5) and the slope of $y = 3x - 2$ as 3, we get

$$\left(\frac{y - 5}{x - 3} = 3\right) \Rightarrow (y - 5 = 3(x - 3)) \Rightarrow (y = 3x - 4).$$

Of course, students should not be expected to master the solution of simultaneous equations by various methods within a 3 day span, and the teacher should provide problems to allow for practice in algebraic manipulations while continuing to teach material in the sections that follow. A natural reenforcement of these concepts does occur in sections 6.16, 6.18 and 6.20 but many of the proofs depend upon an expertise with the solution of pairs of linear equations.

6.15 **Exercises**

1. a. (8, 3)
   b. none
   c. (2, 6)
   d. $(\frac{3}{4}, 3)$
   e. (3, 0)
   f. (3, 4)
   g. $(-\frac{5}{3}, \frac{-26}{3})$
   h. none

2. $\overleftrightarrow{OA}$ has equation $y = x$; $\overleftrightarrow{IJ}$ has equation $x + y = 1$; $\overleftrightarrow{JA}$ has equation $y = \frac{1}{2}x + 1$; $\overleftrightarrow{OI}$ has equation $y = 0$; $\overleftrightarrow{IA}$ has equation $y = 2x - 2$; $\overleftrightarrow{OJ}$ has equation $x = 0$.
   a. The solution of ($y = x$ and $x + y = 1$) is $(\frac{1}{2}, \frac{1}{2})$
   b. The solution of ($y = \frac{1}{2}x + 1$ and $y = 0$) is $(-2, 0)$.
   c. The solution of ($y = 2x - 2$ and $x = 0$) is $(0, -2)$

3. An equation for AB is $y = \frac{1}{2}x + \frac{1}{2}$. $(-3, -1)$ satisfies this equation. Therefore C is on $\overleftrightarrow{AB}$.

4. a. An equation for $\overleftrightarrow{AB}$ is $y = 2x - 5$. $(-2, -9)$ satisfies this equation. Therefore C is on $\overleftrightarrow{AB}$.
   b. An equation for $\overleftrightarrow{DE}$ is $y = -2x + 8$. $(3, 1)$ does not satisfy this equation. Therefore D, E, F are not collinear.
   c. An equation for $\overleftrightarrow{KL}$ is $y = x$. It is satisfied by M. Therefore K, L, M are collinear.
   d. The conditions $a \neq 0$, $b \neq 0$ tell us that P, Q, R are

distinct. An equation for $\overleftrightarrow{PR}$ is $x + y = o$, which is satisfied by (o, o). The points are collinear.

Alternate: An equation for $\overleftrightarrow{PQ}$ is $y = -\frac{b}{a}x$, which is satisfied by point R : $(-a, b)$. The points are collinear.

5. a.

x	-1	0	1	2	3
y	-5	-3	-1	1	3

for $(x_1, y_1) = (2, 1)$ and $(x_2, y_2) = (3, 3)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{3 - 2} = \frac{2}{1} = 2$$

for $(x_1, y_1) = (-1, -5)$ and $(x_2, y_2) = (o, -3)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-5)}{o-(-1)} = \frac{2}{1} = 2$$

The results are the same. Yes.

b. For $(x_1, y_1) = (p, 2p-3)$ and $x_2, y_2) = (q, 2q-3)$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{(2q-3) - (2p-3)}{q - p} = \frac{2q-2p}{q-p} = \frac{2(q-p)}{(q-p)} = 2$$

The sentence completion is: $\frac{y_2-y_1}{x_2-x_1} = 2$.

c. For $(x_1, y_1) = (x_1, ax_1 + b)$, and $(x_2, y_2) = (x_2, ax_2 + b)$:

$$\frac{y_2-y_1}{x_2-x_1} = \frac{(ax_2 + b) - (ax_1 + b)}{x_2 - x_1} = \frac{ax_2-ax_1}{x_2-x_1} =$$

$$\frac{a(x_2-x_1)}{(x_2-x_1)} = a.$$

d. 5; -2; 0; $\frac{1}{2}$.

e. The lines are parallel if their equations have no solution or the lines are parallel if the equations are equivalent. ($y=ax+b$ and $y=ax+c$) have a solution if there exists x for which $ax+b = ax+c$, or $b=c$. If $b \neq c$ there is no solution. If $b = c$, the equations are equivalent and the lines are the same. In either case the lines are parallel.

6. Slope of m is $\frac{3-2}{5-6} = -1$. Using the point-slope form with $(x_1, y_1) = (6, 2)$ we get $y - 2 = -1(x - 6)$. Any equivalent equation to this is acceptable.

7. a. $\frac{-5 -(-2)}{3 - 1} = \frac{-3}{2}$

b. Starting with slope $= -\frac{3}{2}$ and using $(x_1, y_1) = (1, -2)$

$y + 2 = -\frac{3}{2}(x - 1)$ or $3x + 2y + 1 = 0$.

c. $(3\cdot 20 + 2y + 1 = 0) \Rightarrow (y = -30\frac{1}{2})$

d. $(3x + 2\cdot 8 + 1 = 0 \Rightarrow (x = -5\frac{2}{5}$

8. $x = -2$ is an equation for a line with no slope.

$y = 3$ is an equation for a line with zero slope.

6.16 Triangles and Quadrilaterals (Time: 6.16, 6.17 = $1\frac{1}{2}$ - 2 days)

Here, proof is the keystone in examining various properties of triangles and quadrilaterals (primarily parallelograms). The basis for these proofs is the plane coordinate system and the properties discovered or defined with regard to linear equations.

The median of a triangle is introduced and defined. Note that the ratio 2:3 described in the text is the ratio AG:AE where $\overline{AE}$ is a median and G is the point at which the 3 medians of the triangle meet. (See Figure 6.22 in text). It is not uncommon to state that the medians of a triangle intersect to form segments that are in a 2:1 ratio. Then, using the example shown, AG:GE = 2:1.

At least 2 of the proofs to be found in section 6.17 should be demonstrated in class before assigning the remaining proofs for homework. Problem 3 would be excellent in this regard.

6.17 Exercises

1. a) Let the triangle be ABC and (A, B, C) as the base of a plane coordinate system.

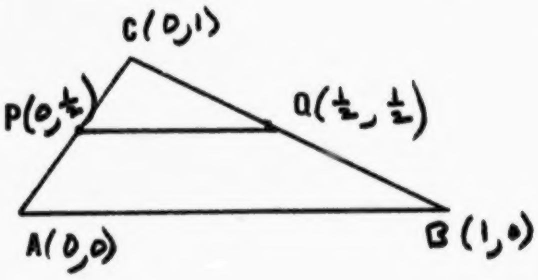
   b) The midpoint of $\overline{AC}$ is $P(0, \frac{1}{2})$ and the midpoint of $\overline{BC}$ os $Q(\frac{1}{2}, \frac{1}{2})$.

   c) $\overleftrightarrow{PQ}$ has the equation $y = \frac{1}{2}$.

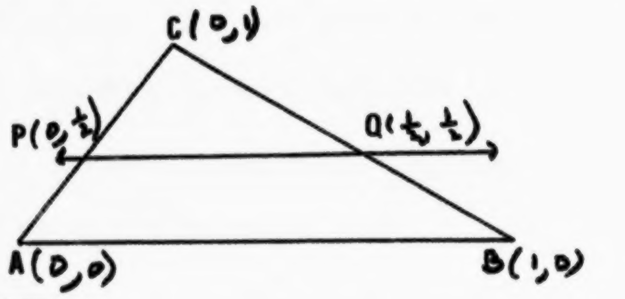
   d) $\overleftrightarrow{AB}$ has the equation $y = 0$.

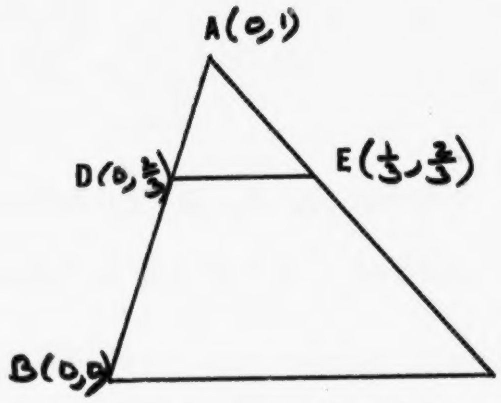
   e) $\overleftrightarrow{AB} \cap \overleftrightarrow{PQ} = \emptyset$

   f) $\overleftrightarrow{AB} \parallel \overleftrightarrow{PQ}$.



2. Let the triangle be ABC and take (A, B, C) as base. $P(0, \frac{1}{2})$ is the midpoint of $\overline{AC}$.
   An equation of ℓ, the line that contains P and is parallel to $\overleftrightarrow{AB}$ is $y = \frac{1}{2}$.
   An equation for $\overleftrightarrow{BC} = x + y = 1$.
   $\ell \cap \overleftrightarrow{BC} = \{Q(\frac{1}{2}, \frac{1}{2})\}$.
   Q is the midpoint of $\overline{BC}$
   ℓ passes through the midpoint of $\overline{BC}$.





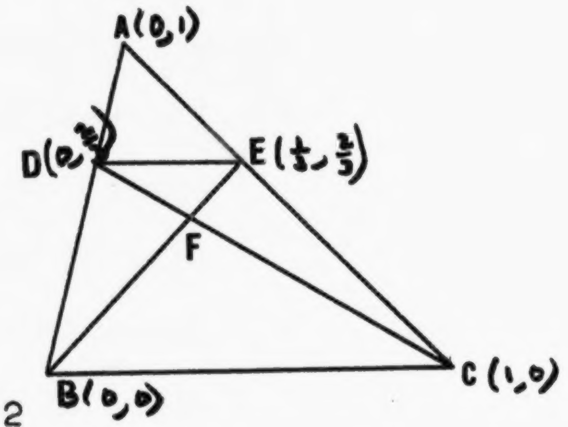
3. (a) Given BD:DA = 2:1 and

CE:EA = 2:1

Prove $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

1. Take (B, C, A) as base given coordinates as indicated for B, C and A.
2. By ratio, D is $\frac{2}{3}$ of the way from B to A and E is $\frac{2}{3}$ of the way from C to A, producing $D(0, \frac{2}{3})$ and $E(\frac{1}{3}, \frac{2}{3})$.
3. $\overleftrightarrow{DE}$ has equation $y = \frac{2}{3}$.

   $\overleftrightarrow{BC}$ has equation $y = 0$.
4. $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.



(b) Given above and $\overline{BE} \cap \overline{CD} = \{F\}$

Prove BF:FE = CF:FD = 3:1

1. $\overleftrightarrow{BE}$ has equation $y = 2x$.
2. $\overleftrightarrow{CD}$ has equation $2x + 3y = 2$

   [from $(\frac{y - 0}{x - 1} = \frac{\frac{2}{3} - 0}{0 - 1}) \Rightarrow (\frac{y}{x - 1} = \frac{\frac{2}{3}}{-1}) \Rightarrow$

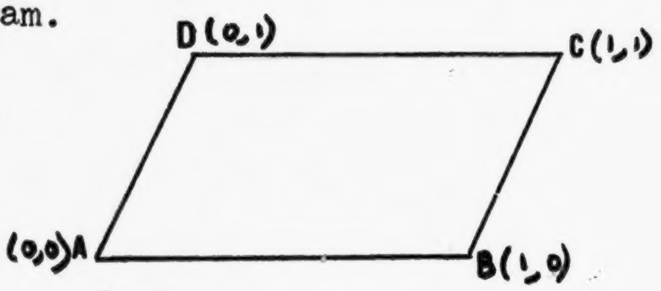
   $(-y = \frac{2}{3}(x - 1)) \Rightarrow (-3y = 2(x - 1)) \Rightarrow$

   $(-3y = 2x - 2) \Rightarrow (2 = 2x + 3y)$]
3. $\overleftrightarrow{BE} \cap \overleftrightarrow{CD} = \{F(\frac{1}{4}, \frac{1}{2})\}$
4. Comparing lengths by distance formula, taking x-coordinates only

   $BF = |0 - \frac{1}{4}| = \frac{1}{4}$; $FE = |\frac{1}{4} - \frac{1}{3}| = |\frac{3}{12} - \frac{4}{12}| = \frac{1}{12}$.
5. Then $BF:FE = \frac{1}{4} : \frac{1}{12} = \frac{3}{12} : \frac{1}{12} = 3:1$
6. Similarly, $CF = |1 - \frac{1}{4}| = \frac{3}{4}$; $FD = |\frac{1}{4} - 0| = \frac{1}{4}$.
7. Then $CF:FD = \frac{3}{4} : \frac{1}{4} = 3:1$.

4. Given A, B, D-coordinate of C is (1, 1).

Prove: ABCD is a parallelogram.



a) Equation for $\overleftrightarrow{AD}$ is $x = 0$;

for $\overleftrightarrow{BC}$ is $x = 1$.

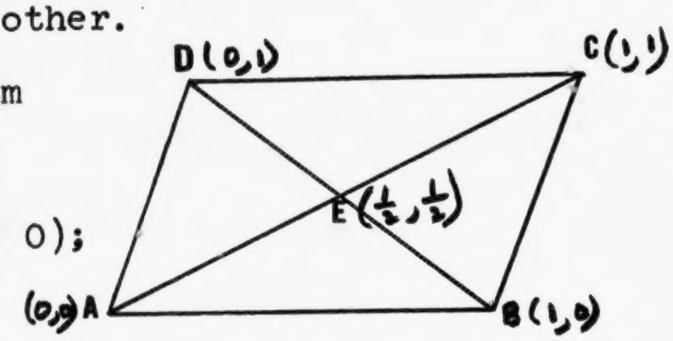
Then $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.

b) Equation for $\overleftrightarrow{AB}$ is $y = 0$; for $\overleftrightarrow{DC}$ is $y = 1$. Then $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$.

c) Since opposite sides are ∥, ABCD is a parallelogram.

5. Given: $\overline{AC}$ and $\overline{BD}$ bisect each other.

Prove: ABCD is a parallelogram



a) Let base be (A, B, D) and assign coordinates to A(0, 0); B(1, 0) and D(0, 1).

b) Since $\overline{BD}$ is bisected, E is midpoint. Then E has coordinates $(\frac{1}{2}, \frac{1}{2})$.

c) Equation for $\overleftrightarrow{AE}$ is $x = y$.

d) Since $\overline{AC}$ is bisected at E, AE = EC. Let C be (x, y)

e) Using only x-coordinates, where AE = EC, we get

$|0 - \frac{1}{2}| = |\frac{1}{2} - x|$ or $\frac{1}{2} = |\frac{1}{2} - x|$ or $x = 1$

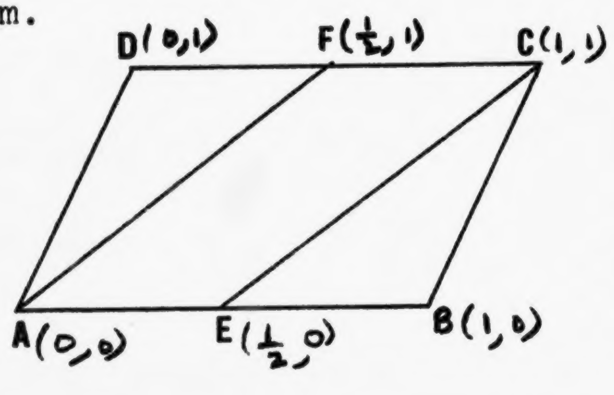
f) Using y-coordinates where AE = EC, we get

$$|0 - \frac{1}{2}| = |\frac{1}{2} - y| \text{ or } y = 1$$

g) Since C has coordinates (1, 1), we know by problem 4 that ABCD is a parallelogram.

6. Given: parallelogram ABCD where E is midpoint of $\overline{AB}$ and F is midpoint of $\overline{DC}$.

Prove: AECF is a parallelogram.



a) Let (A, B, D) be base and assign coordinates to A, B, C, D.

b) By midpoints, E has coordinates $(\frac{1}{2}, 0)$ and $F(\frac{1}{2}, 1)$.

c) Equation for $\overleftrightarrow{AF}$ is $y = 2x$; equation for $\overleftrightarrow{EC}$ is $y = 2x - 1$.

d) Since slopes of $\overleftrightarrow{AF}$ and $\overleftrightarrow{EC}$ are equal, $\overleftrightarrow{AF} \parallel \overleftrightarrow{EC}$.

e) Since $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$ from the given, we know $\overleftrightarrow{FC} \parallel \overleftrightarrow{AE}$.

f) Then, with opposite sides $\parallel$, AECF is a parallelogram.

7. Given: data from no. 6 and

$\overline{AF} \cap \overline{DB} = \{G\}$

Prove: $AG:GF = 2:1$

a) Equation for $\overleftrightarrow{BD}$ is $x + y = 1$;
equation for $\overleftrightarrow{AF}$ is $y = 2x$

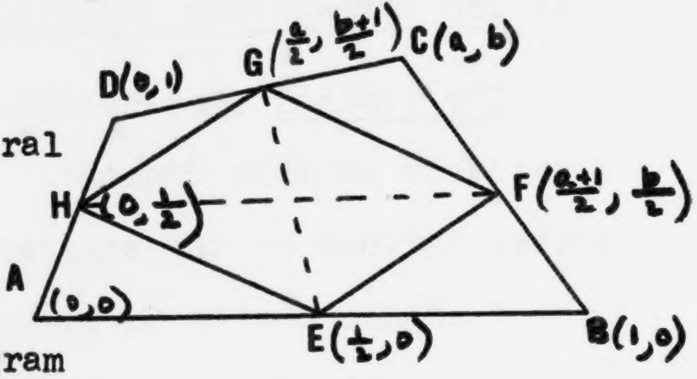
b) $\overleftrightarrow{AF} \cap \overleftrightarrow{BD} = \{G(\frac{1}{3}, \frac{2}{3})\}$

c) Using the x-coordinates for A, G, F as $0, \frac{1}{3}, \frac{1}{2}$, we get

$AG:GF = |0 - \frac{1}{3}| : |\frac{1}{3} - \frac{1}{2}| = \frac{1}{3} : |\frac{2}{6} - \frac{3}{6}| = \frac{1}{3} : \frac{1}{6} = \frac{2}{6} : \frac{1}{6} = 2 : 1.$

8. Given: ABCD is a quadrilateral with midpoints as listed in diagram.

Prove: EFGH is a parallelogram if it is a quadrilateral.



Take (A, B, D) as base of a coordinate system and let C have coordinates (a, b). The midpoint E of $\overline{AB}$ has coordinates $(\frac{1}{2}, 0)$.

The midpoint F of $\overline{BC}$ has coordinates $(\frac{a+1}{2}, \frac{b}{2})$.

The midpoint of G of $\overline{CD}$ has coordinates $(\frac{a}{2}, \frac{b+1}{2})$.

The midpoint H of $\overline{DA}$ has coordinates $(0, \frac{1}{2})$.

The midpoint of $\overline{EG}$ has coordinates $(\frac{a+1}{4}, \frac{b+1}{4})$.

The midpoint of $\overline{FH}$ has coordinates $(\frac{a+1}{4}, \frac{b+1}{4})$.

Therefore $\overline{EG}$ and $\overline{FH}$ bisect each other. By Exercise 5 EFGH, if it is a quadrilateral, it is a parallelogram.

Note.

In order to show a figure is a parallelogram, we must first show it is a quadrilateral. (See definition of parallelogram). To do so here we would have to show that E, F, G, H are four points, no three of which are collinear, that $EF \cap GH = \phi$, and $FG \cap EH = \phi$. (See definition of quadrilateral). But this was not asked for in this problem.

6.18 The Pythagorean Property (Time: 6.18, 6.19 = $1\frac{1}{2}$ - 2 days)

The teacher should note that this topic is not developed rigorously in this Chapter. Any attempt to do so within the system defined by the six axioms faces difficulties which we cannot expect pupils to overcome. There are two major difficulties (and a number of minor ones). The first is to define a perpendicularity relation for lines. A perpendicularity relation may be defined as follows (patterned after Modern Coordinate Geometry, Part 1, page 263).

Definition. A perpendicularity relation in an affine plane is any relation (denoted by $\perp$ and read "is perpendicular to") between two lines in the plane such that

(i) for no line ℓ is $\ell \perp \ell$.

(ii) if $\ell \perp m$, then $m \perp \ell$.

(iii) if P is a point in the plane and ℓ is a line, there is one and only one line m in the plane containing P for which $m \perp \ell$.

With these properties of a perpendicularity relation one may then prove the following basic theorems for lines in the plane.

1. If $\ell_1 \perp m$ and $\ell_2 \perp m$, then $\ell_1 \parallel \ell_2$.
2. If $\ell_1 \perp m$ and $\ell_1 \parallel \ell_2$, then $\ell_2 \perp m$.
3. If $\ell_1 \perp \ell_2$ then it is not true that $\ell_1 \parallel \ell_2$.
4. If $\ell_1 \parallel m_1$ and $\ell_2 \parallel m_2$, then $\ell_1 \perp \ell_2$ if and only if $m_1 \perp m_2$.

The second major difficulty is to define a function, d, which serves to yield distances on different lines that are comparable. A function that has the following properties would be suitable.

(1) d is defined on all ordered pairs of points of the plane.

(2) On any line in the plane d is a linear distance function.

(3) Any parallel projection from a line to a parallel line preserves d.

(4) The Pythagorean property holds.

Having defined d and $\perp$ there is still the problem of proving that these can be introduced into a plane coordinate system. This is a most vexing problem for it can be shown that any plane coordinate system (no matter how the axes are taken) can be transformed into a rectangular coordinate system. For a proof of this assertion see Modern Coordinate Geometry Section 7.5.

The students should have some practice with selected problems from exercises 1, 2, 3 and 4 from section 6.19 before assigning homework.

6.19 Exercises

1. a. 5 d. 12
   b. 17 e. 24
   c. 6 f. 24

2. a. $\sqrt{2}$ e. $\sqrt{21}$ i. $\sqrt{3}$
   b. $\sqrt{5}$ f. $\sqrt{41}$
   c. $\sqrt{3}$ g. 4
   d. $\sqrt{7}$ h. 1

3. a. 5 c. 13 e. $\sqrt{58}$
   b. 5 d. $\sqrt{8}$ or $2\sqrt{2}$ f. $\sqrt{5}$

4. a. (0, 8) c. $(0, \sqrt{8})$ or $(0, 2\sqrt{2})$
   b. (0, 5) d. $(0, \sqrt{108})$ or $(0, 6\sqrt{3})$

5. a. $x = 8$, $y = 25$ b. $y = 15$, $x = 17$

6.20 Plane Rectangular Coordinate Systems

(Time: 6.20, 6.21 = 2-3 days)

Using a plane coordinate system with base (O, I, J), restrictions are made to define a rectangular coordinate system:

a. $\overleftrightarrow{OI} \perp \overleftrightarrow{OJ}$

b. $OI = OJ = 1$

The importance of this system comes in the development of the distance formula for any 2 points in the plane given by coordinates (x_1, y_1) and (x_2, y_2):

$$D = \text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

The relation between this formula and the Pythagorean property $c = \sqrt{a^2 + b^2}$ should be clear and should demonstrate the need for the rectangular coordinate system.

6.21 Exercises

1. a. $AB = 5$
   b. $CD = 5$
   c. $EF = 5$
   d. $PQ = 4$
   e. $RS = \sqrt{40}$
   f. $TV = 0$
   g. $AB = \sqrt{125}$
   h. $CD = \sqrt{13}$
   i. $EF = \sqrt{a^2 + b^2}$
   j. $GH = |b - c|$
   k. $KL = |a - c|$
   l. $MN = \sqrt{(a-c)^2+(b-d)^2}$

2. a. $AB = \sqrt{52} = BC$
   b. $AB = \sqrt{50} = BC$
   c. $AC = \sqrt{85} = BC$
   d. $AB = \sqrt{20} = BC$
   e. $AB = \sqrt{10} = AC$

3. C has coordinates (3, 4). $AC = \sqrt{3^2 + 4^2} = BD$

4. a. Midpoint D of $\overline{AB}$ has coordinates (3, 4).

   $CD = \sqrt{3^2 + 4^2} = 5$, $AB = \sqrt{6^2 + 8^2} = 10 \therefore CD = \frac{1}{2}AB$.

   b. D has coordinates $(6, \frac{5}{2})$.

   $CD = \sqrt{6^2 + (\frac{5}{2})^2} = \sqrt{36 + \frac{25}{4}} = \frac{169}{4} = \frac{13}{2}$.

   $AB = \sqrt{122}$

   $AB = \sqrt{12^2 + 5^2} = 13$

   $\therefore CD = \frac{1}{2}AB$.

d. half the length of the hypotenuse.

5. The midpoint D of $\overline{AB}$ has coordinates (2, 3).
The midpoint E of $\overline{CB}$ has coordinates (-2, 3).
$AE = \sqrt{6^2 + 3^2}$, $CD = \sqrt{6^2 + 3^2}$ $\quad\therefore AE = CD$

6. Slope of $\overleftrightarrow{AB} = \frac{2}{3}$ = slope of $\overline{CD}$ $\quad\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$
Slope of $\overleftrightarrow{BC} = 1$ = slope of $\overleftrightarrow{AD}$ $\quad\therefore \overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$.
$\therefore$ ABCD is a parallelogram.
$AB = \sqrt{3^2 + 2^2} = CD$, $BC = \sqrt{4^2 + 4^2} = DA$.

7. Substituting the distances for AB, AC, and BC in
$(AB)^2 = (AC)^2 + (BC)^2$ we get
$b^2 + a^2 = c^2 + a^2 + c^2 - 2cb + b^2$ which is equivalent to
$0 = 2c^2 - 2cb = 2c(c - b)$.
This implies that either $c = 0$ or $c - b = 0$. The latter implies $c = b$ or $C = B$ and we do not have the triangle ABC that was given. Hence $c = 0$ or $C = D$ and $\angle ACB$ is a right angle.

8. The numbers that can be sides of a right triangle are
in a. (hypotenuse length = 25)
b. (hypotenuse length = 25)
c. (hypotenuse length = 7)
f. (hypotenuse length = 4)
h. (hypotenuse length = 41)
i. (hypotenuse length = 5a).

6.22 Summary

Assign for student reading.

6.23 Review Exercises

1. a. $y = 5x + 2$ b. $y = 2x - 8$

2. a. $x = \frac{1}{5}y - \frac{2}{5}$ b. $x = \frac{1}{2}y + 4$

3. a. $\{P(x) \mid -2 \leq x \leq 1\}$

   b. $\{P(x) \mid x \geq -2\}$

   c. $\{P(x) \mid x \leq 1\}$

   d. $\{P(x) \mid x$ is any real number$\}$.

   e. $\{P(x) \mid -1 \leq x \leq 0\}$

   f. $\{P(x) \mid -2 \leq x \leq 0\}$.

   g. $\{P(x) \mid x \geq -2\}$

   h. $\{P(x) \mid x = 0\}$

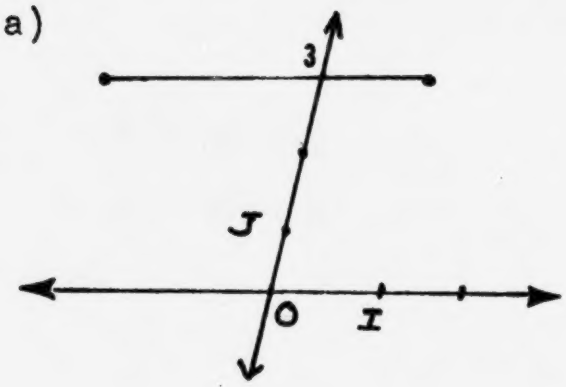
   i. ϕ

   j. $\{P(x) \mid x = \frac{1}{2}\}$.

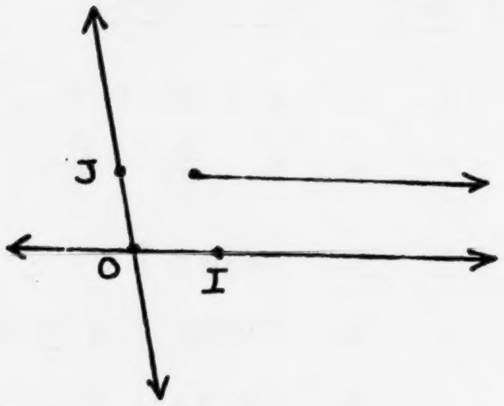
   k. $\{P(x) \mid x = -\frac{2}{7}\}$

4. We can prove that B divides $\overline{AC}$, from A to C, in the same ratio as E divides $\overline{DF}$, from D to F, if we can show that the A, C-coordinate of B is equal to the D, F-coordinate of E. Since A $\xrightarrow{\ell}$ D, B $\xrightarrow{\ell}$ E, C $\xrightarrow{\ell}$ F. This follows directly from Axiom 6. To prove that C' divides $\overline{BA}$, from B to A, in the same ratio as F divides $\overline{ED}$ from E to D, use the B, A- and E, D-coordinate systems and Axiom 6.

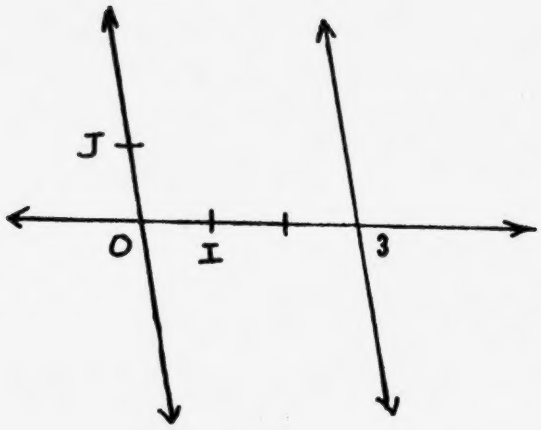
5. a)



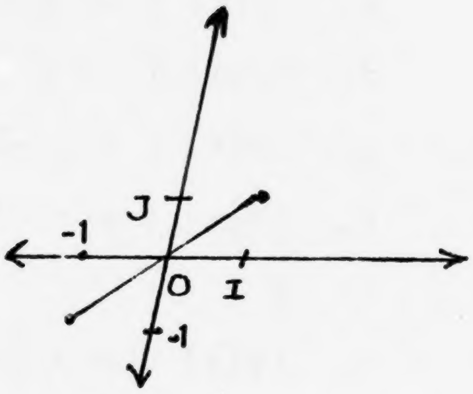
b)



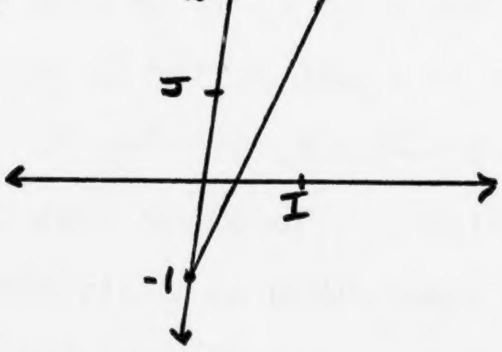
c)



d)



e)



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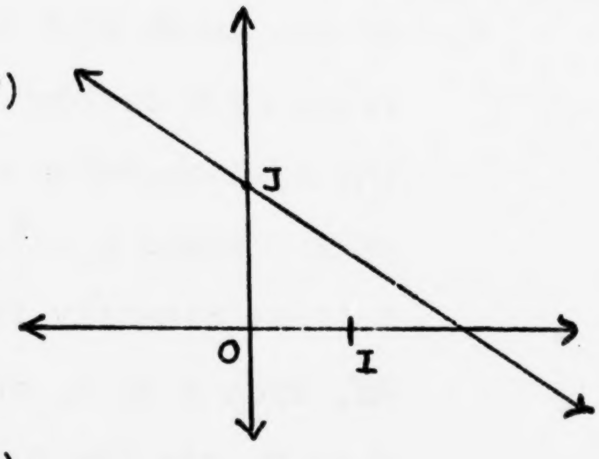

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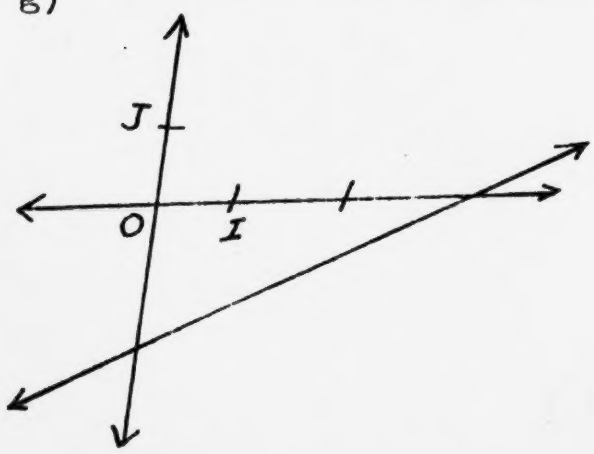
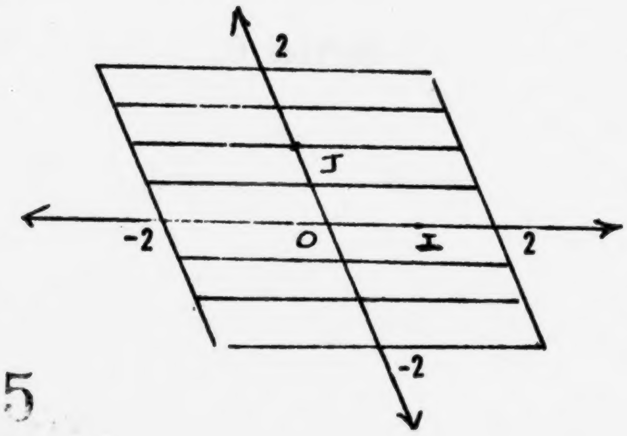


h)



6. a. $y = 2$
   b. $x = -2$
   c. $y = x$
   d. $x + y = 0$
   e. $2x+3y = 6$
   f. $y=-\frac{5}{2}x - 2$

7. a. Midpoint of $\overline{AB}$ has coordinates $(\frac{11}{2}, 2)$. Slope of $\overleftrightarrow{AB} = 0$.

   b. Midpoint of $\overline{DE}$ has coordinates $(0, 0)$. Slope of $\overleftrightarrow{DE} = 1$.

   c. Midpoint of KL has coordinates $(-1, \frac{1}{2})$. Slope of $\overleftrightarrow{KL} = -\frac{5}{2}$.

8. Using (A, B, C) as base of a coordinate system, D has coordinates $(\frac{1}{2}, \frac{1}{2})$ and E has coordinates $(\frac{1}{3}, 0)$. $\overleftrightarrow{AD}$ has equation (1) $y = x$. $\overleftrightarrow{CE}$ has equation (2). $3x + y = 1$. Solving (1) and (2), we get point intersection $P(\frac{1}{4}, \frac{1}{4})$.

   The x-coordinates of A, P, and D, namely $0, \frac{1}{4}, \frac{1}{2}$ are evidence that P bisects $\overline{AD}$.

9. Using (A, B, C) as base of a coordinate system, 1) D has coordinates $(\frac{1}{2}, 0)$ and E has coordinates $(\frac{1}{4}, \frac{1}{2})$. An equation for $\overleftrightarrow{AE}$ is (1)$y = 2x$. An equation for $\overleftrightarrow{CB}$ is (2)$x + y = 1$. Solving (1) and (2) gives $F(\frac{1}{3}, \frac{2}{3})$. The x-coordinates of C, F, B, namely, $0, \frac{1}{3}, 1$ show CF:FB = 1:2.

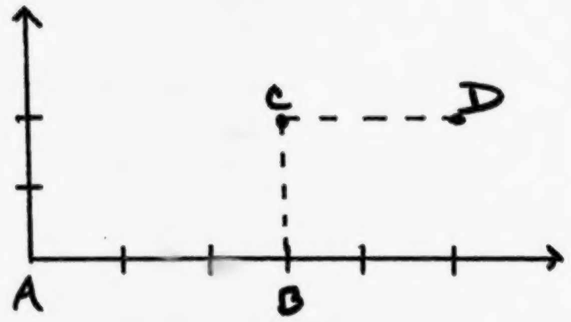
10. Using (A, B, D) as base, C has coordinates (1, 1), E has coordinates $(\frac{2}{3}, 0)$ and F has coordinates $(\frac{1}{3}, 1)$. Slope of $\overleftrightarrow{AF} = 3$. Slope of $\overleftrightarrow{EC} = 3$. Therefore $\overleftrightarrow{AF} \parallel \overleftrightarrow{EC}$ and AECF is a parallelogram. Thus $\overline{AC}$ and $\overline{FE}$ bisect each other, that is, have a common midpoint. But $\overline{DB}$ and $\overline{AC}$ also have a common midpoint. Therefore $\overline{AC}$, $\overline{BD}$, and $\overline{FE}$ meet in a

point.

11. a. AB = 50
    b. AB = $\sqrt{13}$
    c. AC = $\sqrt{39}$
    d. BC = $\sqrt{119}$.

12. a. AB = 6
    b. CD = 16
    c. EF = 5
    d. GH = $\sqrt{125}$

13. Using the rectangular coordinate system, indicated at the right, the final position D has coordinates (5, 2).

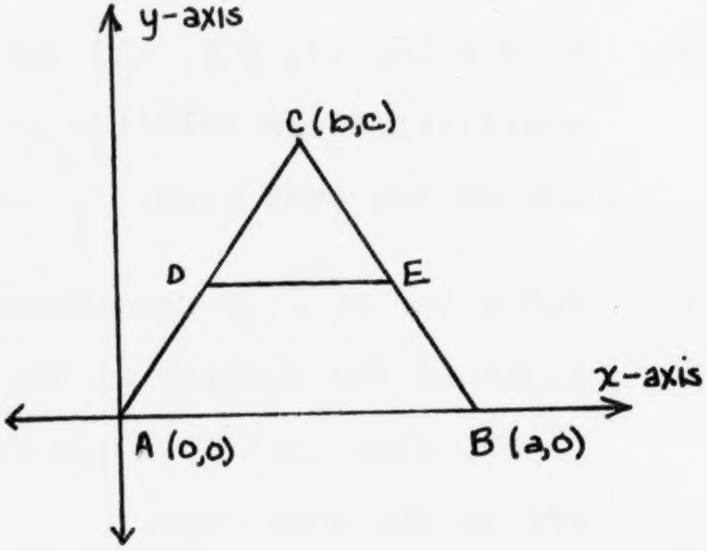


AD = $\sqrt{25 + 4} = \sqrt{29}$ miles.

14. The midpoint of $\overline{AC}$ has coordinates (4, 3).
The midpoint of BD has coordinates (4, 3)
∴ ABCD is a parallelogram.
E has coordinates (1, 2); F has coordinates (5, 5); G has coordinates (7, 4); and H has coordinates (3, 1).

EF = $\sqrt{4^2 + 3^2} = 5$
FG = $\sqrt{2^2 + 1^2} = \sqrt{5}$
GH = $\sqrt{4^2 + 3^2} = 5$
HE = $\sqrt{2^2 + 1^2} = \sqrt{5}$
∴ EF = GH and FG = HE.

15. To prove lengths on different lines equal we must use rectangular coordinate systems. Let the triangle be ABC. Let AB be the positive x-axis of a rectangular coordinate system, let B have coordinates (a, o) and



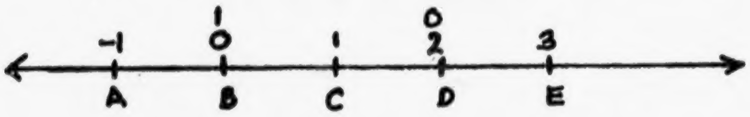
let C have coordinates (b,c). Then midpoint D of $\overline{AC}$ has coordinates $(\frac{b}{2}, \frac{c}{2})$ and E the midpoint of $\overline{CB}$ has coordinates $(\frac{a+b}{2}, \frac{c}{2})$. Using the distance formula twice, AB = $\sqrt{a^2}$, DE = $\sqrt{\frac{a^2}{4}} = \sqrt{\frac{a}{2}^2}$.

Or simply AB = a $\therefore$ DE = $\frac{1}{2}$ AB.

DE = $\frac{a}{2}$

Suggestions for Test Items for Chapter 6 Course II.

1. Assume two coordinate systems on the line below, one of which assigns coordinate x to a point and the other assigns x' to the same point.



x' - coordinates

x - coordinates

a. What is the B, C-coordinate of the midpoint of $\overline{BC}$?

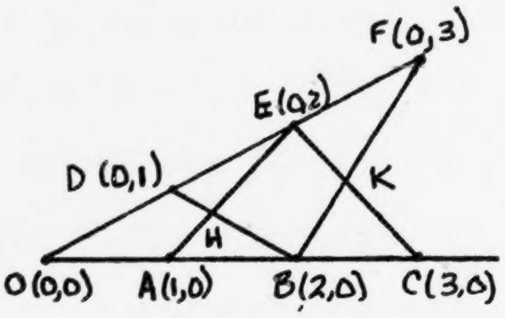
b. What is B, D-coordinate of A?

c. What is the formula that converts x-coordinates to x'-coordinates? Give answer in the form $x' = ax + b$.

d. What is the formula that converts x'-coordinates to x-coordinates? Again give answer in the form $x = ax' + b$.

4. Show A (6, 3), B(2, -3) and C(100, 144), where all coordinates are relative to the same coordinate systems, are on the same line.

5. Using the O, A, D- coordinates given in the diagram at the right, show that O, H and K are on the same line.

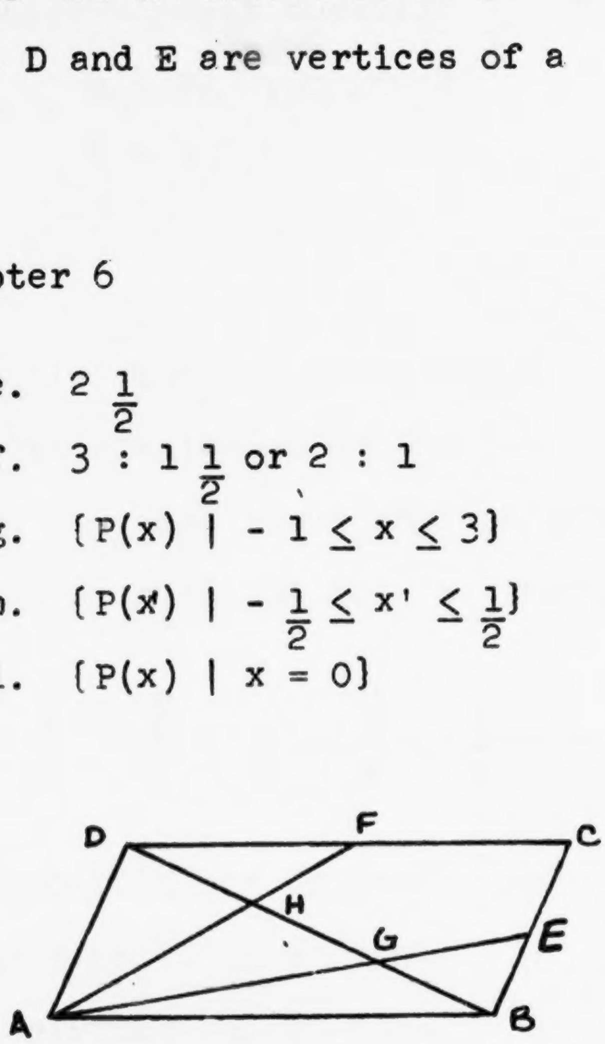


6. In triangle ABC let medians $\overline{AD}$ and $\overline{BE}$ intersect at G. Show that the midpoints of $\overline{AG}$, $\overline{BG}$, D and E are vertices of a parallelogram.

Answers for Test Items - Chapter 6

1. a. $\frac{1}{2}$
   b. $-\frac{1}{2}$
   c. $x' = -\frac{1}{2}x + 1$
   d. $x = -2x' + 2$
   e. $2\frac{1}{2}$
   f. $3 : 1\frac{1}{2}$ or $2 : 1$
   g. $\{P(x) \mid -1 \leq x \leq 3\}$
   h. $\{P(x') \mid -\frac{1}{2} \leq x' \leq \frac{1}{2}\}$
   i. $\{P(x) \mid x = 0\}$

2. Using (A, B, D) as base, the coordinates of C are (1, 1) of E $(1, \frac{1}{2})$, of F $(\frac{1}{2}, 1)$. An equation for $\overleftrightarrow{BD}$ is (1) $x + y = 1$: for $\overleftrightarrow{AE}$: (2) $y = \frac{1}{2}x$, for $\overleftrightarrow{AF}$: (3) $y = 2x$.



Solving (1) and (2) shows coordinates of G to be $(\frac{2}{3}, \frac{1}{3})$.
Solving (1) and (3) shows coordinates of H to be $(\frac{1}{3}, \frac{2}{3})$.
Since the coordinates of D, H, G, B are respectively $0, \frac{1}{3}, \frac{2}{3}, 1$ it follows that DH=HG=GB.

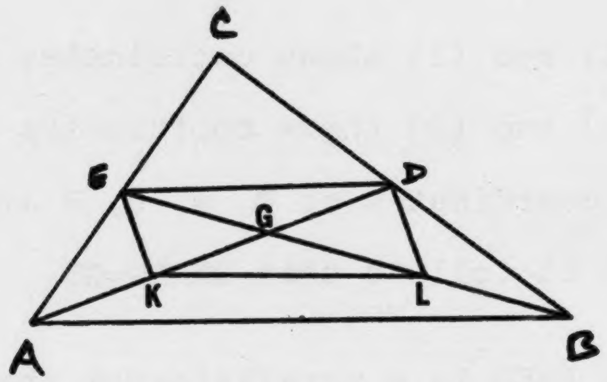
3. (a) Since ABED is a parallelogram the A, D, B- coordinates of E are (1, 1). Since $\overleftrightarrow{CF} \parallel \overleftrightarrow{AD}$ all y-coordinates of points on CF are b. $\therefore$ represent F coordinate as (x,b).
   (b) Slope of $\overleftrightarrow{BC} = \frac{b-1}{a}$ = slope of $\overleftrightarrow{EF} = \frac{b-1}{x-1}$ $\therefore$ $x-1 = a$ or $x=a+1$.
   (c) Slope of $\overleftrightarrow{AC} = \frac{b}{a}$; slope of $\overleftrightarrow{DF} = \frac{b}{a}$. $\therefore$ $\overleftrightarrow{AC} \parallel \overleftrightarrow{DF}$.

4. Slope of $\overleftrightarrow{AB} = \frac{6}{4}$ or $\frac{3}{2}$.
Equation of $\overleftrightarrow{AB} = y-3 = \frac{3}{2}(x-6)$. This equation is satisfied by (100, 144), since $144 - 3 = \frac{3}{2}(100 - 6)$. Therefore A, B, C are on the same line.

5. An equation for $\overleftrightarrow{AE}$ is (1) $2x + y = 2$.
An equation for $\overleftrightarrow{DB}$ is (2) $x + 2y = 2$.
H has coordinates $(\frac{2}{3}, \frac{2}{3})$, the solution of (1) and (2).
An equation for $\overleftrightarrow{EC}$ is (3) $2x + 3y = 6$.
An equation for $\overleftrightarrow{BF}$ is (4) $3x + 2y = 6$.
K has coordinates $(\frac{6}{5}, \frac{6}{5})$, the solution of (3) and (4).
The coordinator of O, H, K satisfy $y = x$. Therefore O, H, K are collinear.

6. Let the midpoint of $\overline{AG}$ be K and let the midpoint of $\overline{BG}$ be L. Since the medians meet in a point that divides each median, from vertex to midpoint, in ratio 2 : 3, AK=KG=GD and BL=LG=GE. Since $\overline{KD}$ and $\overline{LE}$ bisect each other KLDE is a

parallelogram.



CHAPTER 7

REAL FUNCTIONS

Time Estimate For Chapter: 14 days

This chapter has three main objectives: (1) Review and extension of the mapping concept, (2) Examination of several basic mappings whose domain and range set are subsets of the real numbers--using both algebraic and graphic methods--and (3) Introduction to the operations of addition, multiplication, and composition of real functions.

Formally, a mapping is an ordered triple (S,T,h) where S is a set, T a set, and "h" represents a process which assigns to each element s of S (the domain) one and only one element t of T (the codomain) called the image of s. For brevity, mappings are often named with a single letter "f," "g," "h," etc. However, it is critical that the domain and codomain be clearly understood in each discussion of a mapping. A mapping is one-to-one if and only if no element of the codomain serves as the image of more than one element of the domain. A mapping is onto its codomain if and only if each element of the codomain serves as the image of some element of the domain.

A mapping (S,T,h) is indicated

$$h: S \longrightarrow T$$

and individual assignments are indicated

$$s \xrightarrow{h} t \quad \text{or} \quad h(s) = t.$$

Because it is commonly used by mathematicians, the term "real function" is introduced to refer to mappings with domain and codomain some subset of R, the real numbers. The word "function" *is synonymous* with the word "mapping"!

The chapter should be completed in 14 class days.

7.1 Mathematical Mappings (1 day)

The purpose of this section is review of the mapping concept and notation.

Answers to questions in the text:

First set: (1) Domain is all postal addresses in U.S.

(2) Codomain could be any set of numbers which contain the whole numbers like 54494, 10027, 07639, etc.

Second set: (1) Yes. Construct a line through the point of ℓ_1 parallel to ℓ_3. Take as image the intersection with ℓ_2.

(2) Yes. Same process as in (1) reversed.

7.2 Exercises

1. (a) does. (Each element of S is assigned one and only one element of T).

   (b) does. (Unless your school has another grading system).

   (c) does not--*a* has two images.

   (d) does not--o has no image

   (Note that f could easily be restricted to a mapping by changing domain to $Q \setminus \{o\}$. *But* this then is *a* different situation)

(e) does not--2 has two images.

2. (a) -10 (b) $\frac{2}{5}$ (c) 14 (d) $\frac{18}{5}$ (e) 2
3. (a) 4 (b) -4 (c) $\frac{-11}{2}$ (d) 154 (e) -2
4. (a) -12 (b) $\frac{-34}{5}$ (c) 0 (d) $\frac{-26}{5}$ (e) -6
(a) 24 (b) 8 (c) 5 (d) 324 (e) 12
5. (a) 12 (b) $\frac{8}{5}$ (c) 12 (d) $\frac{8}{5}$ (e) 0
(a) ± 6 (b) ___ (c) __ (d) ± 156 (e) 0

7.3 Properties of Real Functions (1 - 1½ days)

The function notation $f(3) = 9$ has been mentioned briefly in chapter 2, so it should be slightly familiar. The phraseology "f of 3 equals 9" is a shortening of the more correct "f assigns 9 as the image of 3" or "the image of 3 under f is 9." The short form might seem unnatural to students--in any case, frequent use of the longer more more correct phrase will help and also keep the meaning of the notation straight.

The assignment processes $x \longrightarrow x^2$ and $x \longrightarrow |x|$ do not define one-to-one functions of R onto R. Here the importance of determining domain and codomain clearly can be stressed because both processes will give one-to-one function of $R_{\geq 0}$ onto $R_{\geq 0}$. Sometimes these restricted domain and codomains will suffice for problems to be solved, often not.

Answers to questions in text:

First set: (1) 0, 2, 7

(2) $\pm \frac{2}{3}$

Second set: -2 has no pre-image under $x \xrightarrow{h} |x|$

Third set: (1) Yes.

(2) No.

(3) No.

7.4 Exercises

1. (a) (i) 2, (ii) 7, (iii) 7, (iv) $9\frac{1}{2}$, (v) 29, (vi) 634.

(b) (i) ±8, (ii) __, (iii) ±2, (iv) __, (v) __, (vi) 0.

(c) $\{x \in R: x \geq 2\}$

(d) No, -2 not an image

(e) No, $f(2) = 4$ and $f(-2) = 4$

(f) No, $f(2) = f(-2) = 4$

2. (a) (i) 0, (ii) $-7\frac{1}{3}$, (iii) 2, (iv) -5 (v) $-\sqrt{2}$, (vi) π

(b) Yes, if $g(a) = g(b)$, then $-a = -b \longrightarrow a = b$

(c) R

(d) Yes, each element of R has an opposite and for each $x \in R$, $-x \xrightarrow{g} x$.

(e) 0 (No others since $x = -x \longrightarrow x + x = 0 \rightarrow 2x = 0 \rightarrow x = 0$).

3. (a) (i) -1, (ii) 0, (iii) 0, (iv) 0, (v) 2.

(b) (i) -2, (ii) 3, (iii) ±1, 0.

(c) No, $f(-1) = f(0) = f(1) = 0$.

(d) $\{-2, -1, 0, 1, 2\}$.

(e) No, range $\neq$ codomain.

4. (a) No, many addresses have same zip code.

(b) No, 752,683 is not a zip code.

Range of z is set of numbers like 54494, 73469, 19927, etc. (might not be all 5 digit numbers though. So here is a good example of a range which does not have a neat defining condition.)

5. (a) (i) 1, (ii) $\frac{4}{5}$, (iii) $\frac{2}{3}$, (iv) $\frac{4}{7}$, (v) $\frac{8}{9}$, (vi) $\frac{8}{15}$, (vii) $\frac{8}{13}$, (viii) $\frac{1}{2}$.

(b) (i) $\frac{4}{3}$, (ii) $\frac{3}{2}$, (iii) __, (iv) 1, (v) __, (vi) $\frac{222}{221}$.

(c) Yes, $\frac{1}{x} = \frac{1}{y} \rightarrow \frac{xy}{x} = \frac{xy}{y} \longrightarrow y = x$.

$f(a) = f(b) \longrightarrow \frac{1}{a} = \frac{1}{b} = \frac{ab}{a} = \frac{ab}{b}$ therefore, $b = a$.

(d) $\{x \in R: \frac{1}{2} \leq x \leq 1\}$.

(e) No, $\frac{1}{3} \notin$ range but $\frac{1}{3} \in$ codomain.

6. (a) Yes.

(b) Yes. For each $x \in R$, $-3x + 6 \xrightarrow{g} x$.

7. (a) True.

(b) False.

(c) True.

(d) False.

7.5 Representing Real Functions (1½ - 2 days)

To get started toward graphical representation of a function, it is shown that every real function determines a set of ordered pairs of real numbers. When the points these pairs name are located on a coordinatized plane, the resultant graph gives a general picture of the behavior of the function: (1) where it assigns positive number as images, where negative, and where zero; (2) where it assigns greatest numbers as images and where smallest; (3) which numbers are assigned as images and which are not; etc.

Perpendicular axes are used exclusively, but scales are not always the same on the two axes--partly for convenience and partly because the domain and codomain in applications are very often differently scaled. For example, pressure might be given in pounds per square inch and temperature in degrees centigrade.

After locating many points in the graph of $x \longrightarrow x^2$, the graph is "completed as the pattern indicates." Caution that the function might not behave so nicely. There will be exercises to emphasize this point. Only experience, judgment of the rule, and careful and patient checking will allow one to know when a graph "continues the pattern."

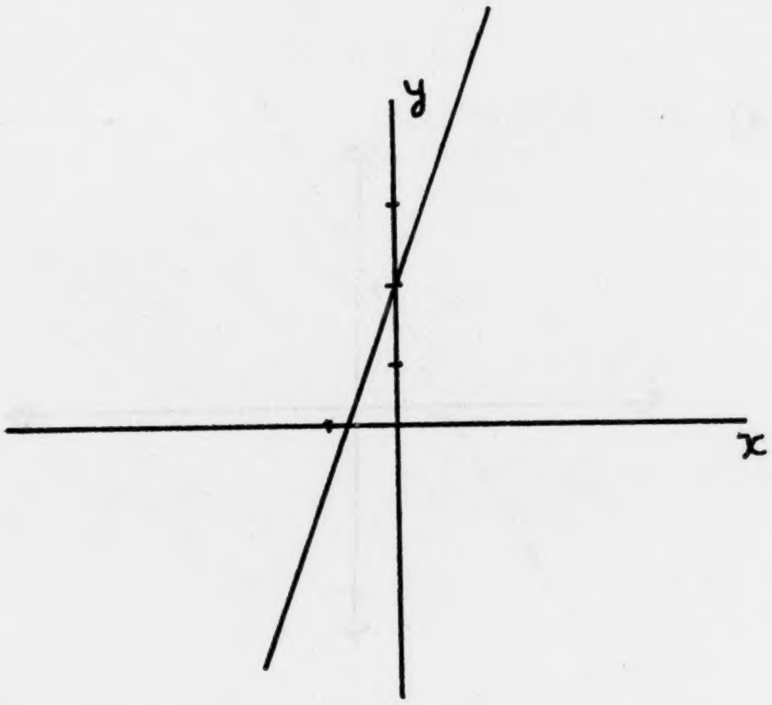
7.6 Exercises

Note: Use of a graph of x^2 to compute $\sqrt{x}$ is dangerous unless the graph is constructed with utmost care.

1. (a) (1.4, 2) (b) (2.2, 5) (c) (2.4, 6) (d) (-2.4, 6) (e) (-2, 4)
2. (a) 1.4 (b) 2.2 (c) 2.4 (d) 2.6 (Answers might vary here).
3. Answers will vary.
4. (a)

0	$-\frac{2}{3}$	3	-3
2	0	11	-7

(b)-(d)

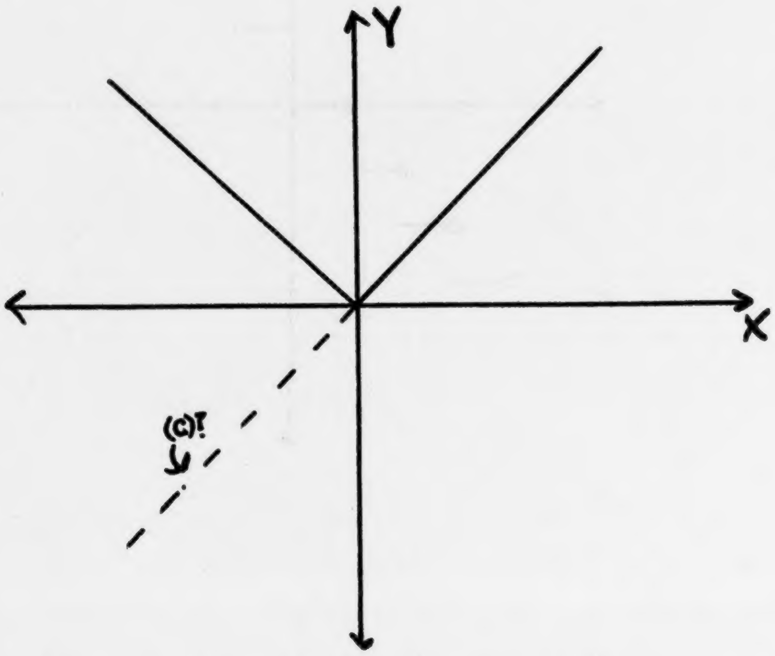


(c) 17, -13, 3

5. (a)

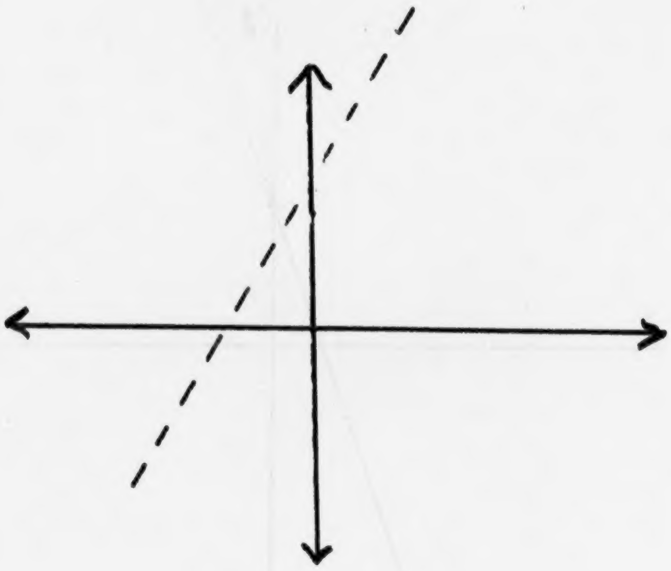
0	2	4	6
0	2	4	6

(b) - (e)



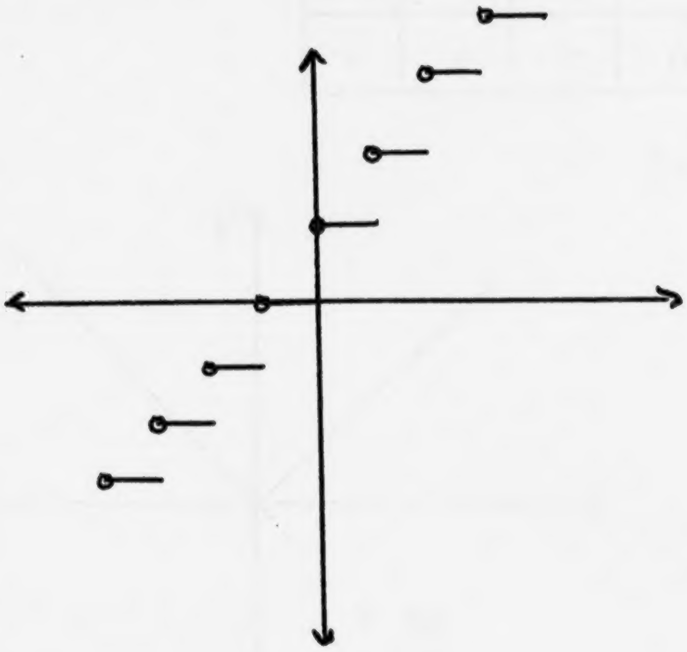
6. (a) -4 -3 -2 -1 0 1 2 3 4 5

(b) and (c)



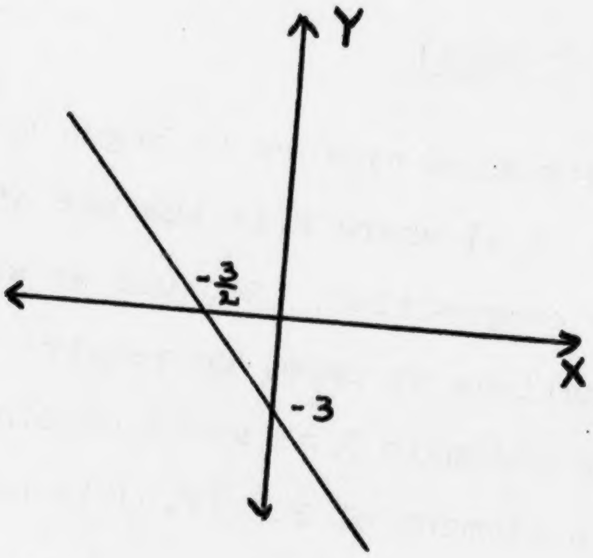
(d) (i) 1, (ii) 1, (iii) 1, (iv) 0, (v) 0,
(vi) 2, (vii) 2, (viii) -2.

(e) The actual graph of p should look like

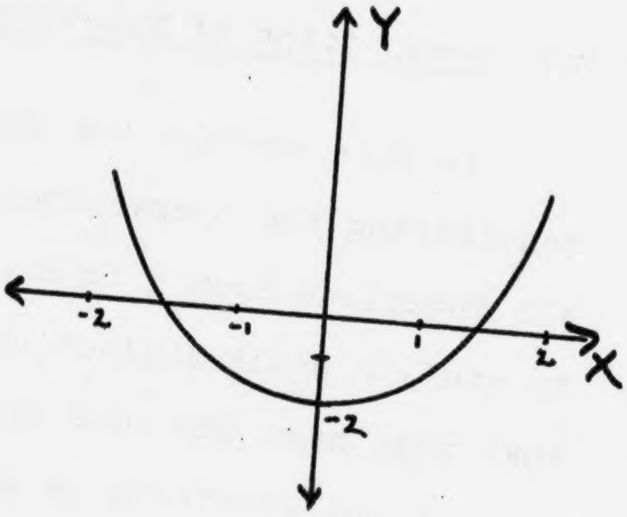


where "o—" indicates the interval (a, b] where a does not belong but b does.

7.



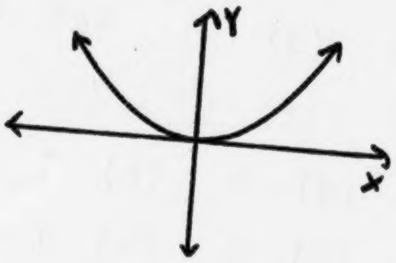
8.



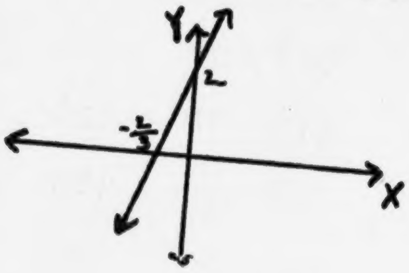
9. (a) no (b) yes (c) no (d) yes.

10. No vertical line intersects the graph in more than one point.

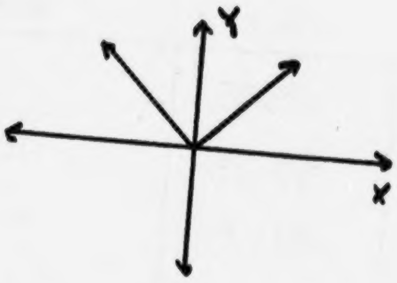
11. (a) no.



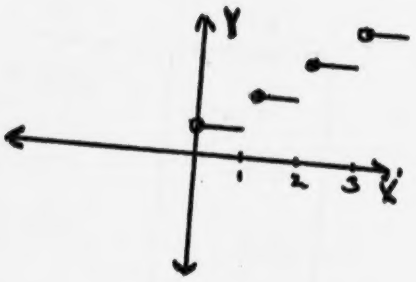
(b) yes.



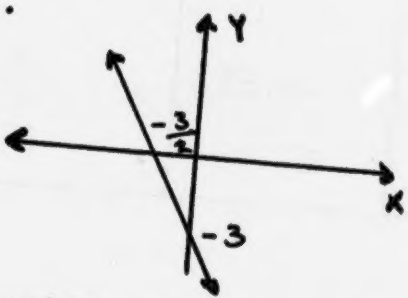
(c) no.



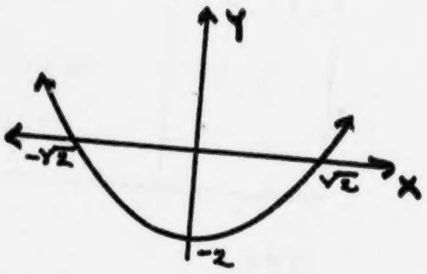
(d) no.



(e) yes.



(f) no.



12. No horizontal line intersects the graph in more than one point.

13. (a) no (b) yes (c) yes (d) no.

7.7 Composition of Functions (1 - 1½ days)

In this section the study of function algebra is begun by considering the operational system (F,o) where F is the set of all functions from R to R and o is composition. Because we want to study a whole collection of functions at once, we require that they have the same domain and codomain R to avoid continual worry about existence of gof as an element of F. (F,o) is not a group--a fact which comes out in Section 7.9.

7.8 Exercises

1. (a) 90 (b) 45 (c) 99.9 (d) 81 (e) 49.2
(f) 30 (g) 15 (h) 33.3 (i) 27 (j) 16.4

2. (a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) $\frac{49}{64}$ (d) 2 (e) 1 (f) 1
(g) 3 (h) 1 (i) 1 (j) 4 (k) 1 (l) 1

3.

x	h	k	hok	koh
0	-24.5	15.75	-8.75	-8.75
19	-5.5	34.75	10.25	10.25
-33	-57.5	-17.25	-41.75	-41.75
-17.25	-41.75	-1.5	-26.0	-26.0
3.14	-21.36	18.89	-5.61	-5.61
-2.7	-27.2	13.05	-11.45	-11.45

Yes

4.

x	m	n	mon	nom
0	0	-7	-126	-7
43	774	36	648	767
-15	-270	-22	-396	-277
12	216	5	90	209

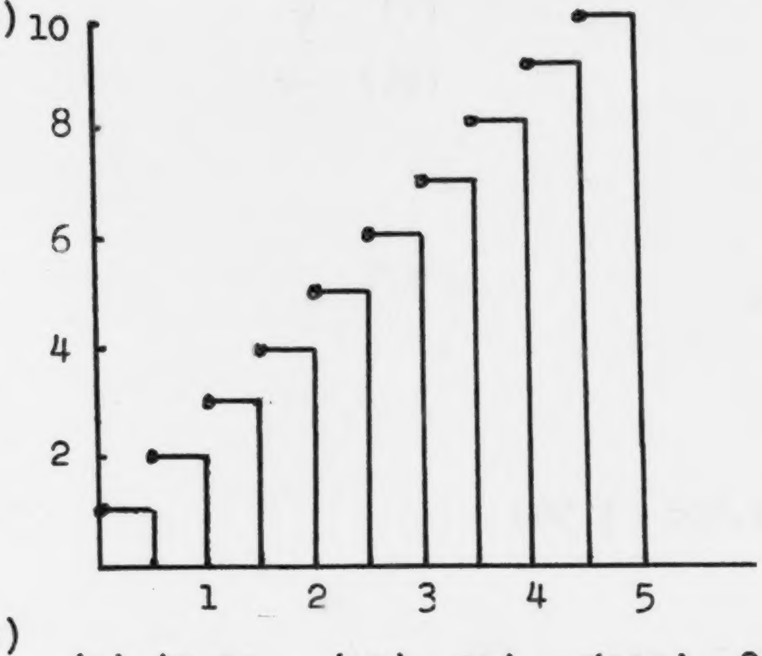
No

5. No (See 4)

6. (a) 6¢ (b) 30¢ (c) 24¢ (d) 6¢

7. (a)
   (i) 14, (ii) 20, (iii) 8, (iv) 1,
   (v) 3, (vi) 5, (vii) 4, (viii) 6.

(b)

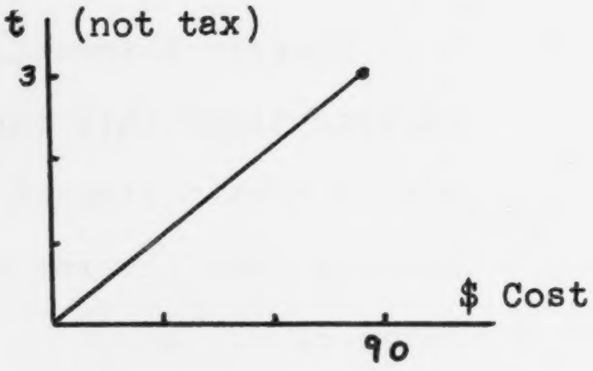


(c)
   (i) $1.10 (ii) 20¢ (iii) 80¢

(d) (2)

8. (a) $ .15 (1)
   $ .13 (2)
   $ .52 (3)
   $3.00 (4)

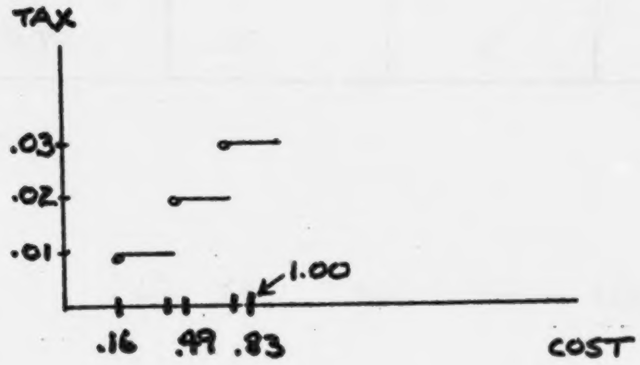
(b)



9. (a) (i) 96.00
(ii) 32.10
(iii) 762.01

(b) (i) 0
(ii) 0
(iii) .01
(iv) .01
(v) .02
(vi) .02
(vii) .03

(c)



RESEMBLES POSTAL FUNCTION

10. (a) (i) 2
(ii) 7
(iii) -4
(iv) 0
(v) $\frac{5}{3}$
(vi) -2

(b) (i) 0
(ii) $\frac{5}{3}$
(iii) -2

(c) $x \xrightarrow{gof} x$

(d) $x \xrightarrow{fog} x$

11. $n \longrightarrow \$.50n + \$.40n = \$.90n$

7.9 Inverses of Real Functions (2 days)

Clearly a function which is not one-to-one can have no inverse since this inverse would have to assign two images to a single domain element. $x \xrightarrow{h} |x|$ from R to R has no inverse because that inverse would be required to assign -2 and 2 as the image of 2.

It is not so clear that there are one-to-one functions without inverses, that is, a function must be *both* one-to-one and onto to have an inverse. The function $x \xrightarrow{f} \frac{1}{|x| + 1}$ maps R one-to-one onto (-1, 1), but not onto R. There are many functions which reverse the assignments of f and thus satisfy

$$g \circ f = j_R,$$

but none which also satisfy

$$f \circ g = j_R,$$

required to make (F,o) a group. The work here becomes ticklish and somewhat messy because we are carrying around R as a codomain.

We are, however, able to rescue the situation in terms of the more general *inverse of a function* as contrasted with the *inverse of an element* in (F,o). Of course, when $f \in F$ has an operational inverse, it coincides with the inverse of f as a function. The proof which follows the definition of a function inverse may be reasonably soft-pedaled in the sense that it observes generally what has been noted several times in examples, both in Course I and in Course II.

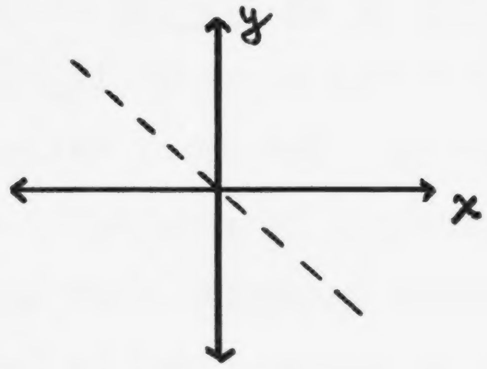
The notion of equivalent functions is introduced here to recognize the fact that in many situations, the range is more important than the entire codomain. This may seem more a terminological than an actual problem. In any case, *equivalent function* is not a central notion at this point, except to enable us to say that a one-to-one function is always equivalent to a *function* having an inverse. In this and all later function algebra, it will be important to remember that $f = g \iff f(x) = g(x)\ \forall\ x$ in domain f, and the domain and codomain of f and g are identical.

7.10 Exercises

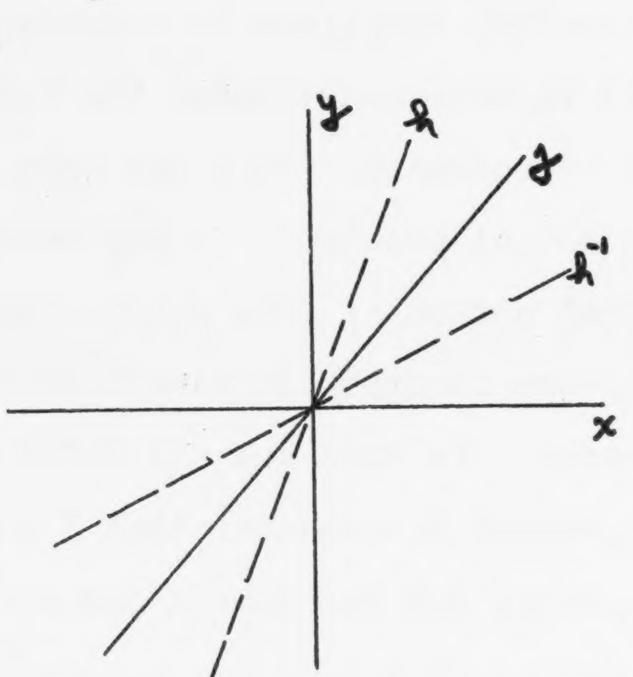
1. (a) not one-to-one, onto, but no inverse.
   (b) one-to-one, not onto, no inverse.
   (c) one-to-one, onto, inverse.
   (d) not one-to-one, onto, no inverse.
   (e) one-to-one, not onto, no inverse.

2. (b) {b', c', d', e'}
   (e) {10, 11, 13, 16, 20, 25, 31, 38}

3. (a) Yes - Each image has a unique pre-image.
   (b) Yes - Codomain equals range.
   (c) Yes $x \xrightarrow{g^{-1}} -x$



4. (a) $x \xrightarrow{h^{-1}} \frac{1}{3}x$
   (b) - (c)

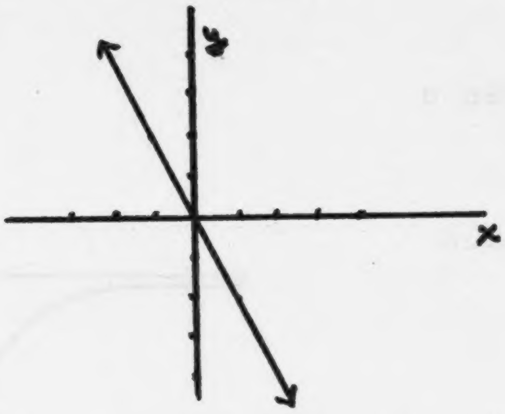


5. (a) Yes, each image has a unique pre-image.

   (b) Yes, codomain equals range.

   (c) Yes $x \xrightarrow{g^{-1}} 2x$

6.



   (a) Yes, each image has a unique pre-image.

   (b) Yes, codomain is equal to the range.

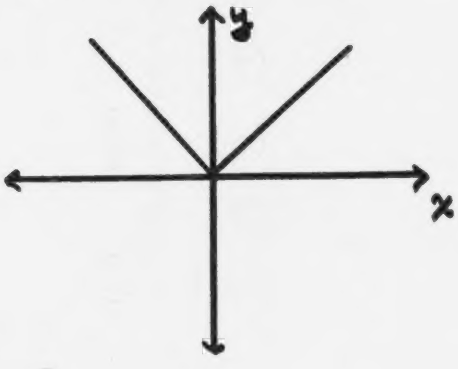
   (c) Yes - $x \xrightarrow{f^{-1}} -\frac{1}{2}x$

7. j bisects angle between f and f^{-1} of f is image of f^{-1} under reflection in j.

8. (a) No, not one-to-one

   (b) -----

   (c) $R_{\geq 0}$ for a domain and codomain.



9. (a) No $f(\frac{1}{2}) = f(1)$

   (b) No $f(x) \overset{?}{=} 0$

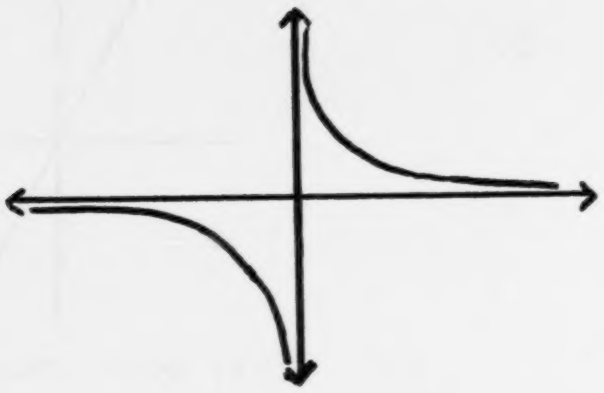
   (c) No not 1 - 1

   (d) By picking one point in each interval (0, 1), (1, 2), etc. for domain.

10. (a) $\frac{1}{0}$ not defined.

(b) $1, \frac{1}{2}, \frac{1}{3}, 2, \frac{2}{3}, \frac{2}{5}, -1, -\frac{1}{2}, -\frac{1}{3}, -2, -\frac{2}{3}, -\frac{2}{5}$.

(c)



(d) Yes, each image has a unique pre-image, Yes, codomain equals range.

(e) Yes

$x \xrightarrow{r^{-1}} \frac{1}{x}$

domain $R \setminus \{0\}$

11. (a)
(1) 0
(2) $\frac{1}{2}$
(3) $-\frac{1}{2}$
(4) $\frac{2}{3}$
(5) $-\frac{2}{3}$
(6) 0
(7) 1
(8) -1
(9) 2
(10) -2

(b)
(1) 0
(2) 1
(3) -1
(4) 2
(5) -2
(6) 0
(7) $\frac{1}{2}$
(8) $-\frac{1}{2}$
(9) $\frac{2}{3}$
(10) $-\frac{2}{3}$

(c) Yes

(d) Yes

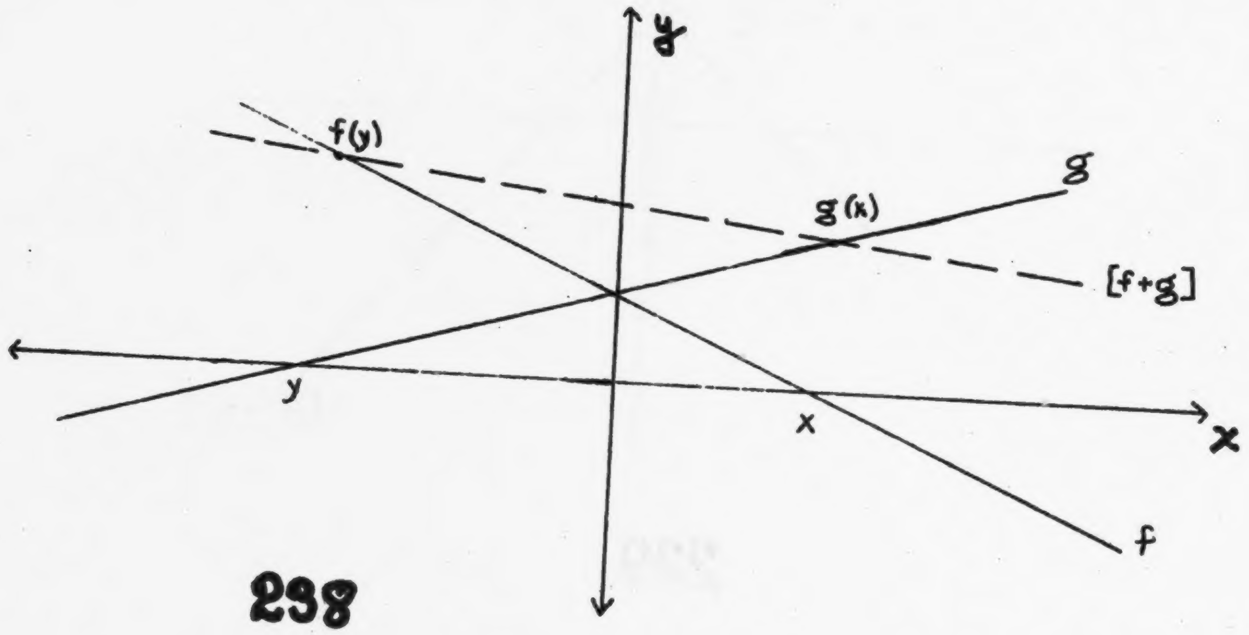
(e) No

(f) No

7.11 [f + g] and [f - g] ($1\frac{1}{2}$ - 2 days)

This section introduces two new operations on functions--operations which are definable primarily because their domain and codomain is the real numbers. The way in which the function [f + g] assigns images is wholly dependent on the possibility of adding any two real numbers. Similar remarks are appropriate for [f - g] and [f · g] and $[\frac{f}{g}]$. It is not possible to define [f - g] on the set of all functions from W to W because f(x) - g(x) might not always name a whole number.

It is most important to emphasize that addition, subtraction, multiplication, and division of functions are operations on F distinct from composition--just as maximizing, lcm, and other operations were defined on W, distinct from + and ·. If f and g are linear and have been graphed, the graphs of [f + g] and [f - g] are obtained by examining the points where f and g are 0. For instance, if f(x) = 0, then [f + g](x) = g(x) and if g(y) = 0, then [f + g](y) = f(y) ∴ (x, g(x)) and (y, f(y)) are two points in the graph of [f + g] and the linear graph of [f + g] is determined.



7.12 Exercises

Note: Exercise 9 should be covered. Its result is used later.

1. (a) (i) 3 (ii) 3 (iii) 3 (iv) 3 (v) 3

(vi) 0 (vii) 1 (viii) 1 (ix) $16\frac{1}{2}$ (x) +23

(b) (i) 3 (ii) 4 (iii) 4 (iv) $19\frac{1}{2}$ (v) 26

(vi) 3 (vii) 2 (viii) 2 (ix) $- 13\frac{1}{2}$ (x) 20

(c) (i) 3 (ii) 3 (iii) 3 (iv) 3 (v) 3

(vi) 3 (vii) 3 (viii) 3 (ix) 3 (x) 3

(d) (i) 3 (ii) 4 (iii) 4 (iv) $19\frac{1}{2}$ (v) 26

(vi) -3 (vii) -2 (viii) -2 (ix) $13\frac{1}{2}$ (x) 20

(e) (i) x $|x| + 3$

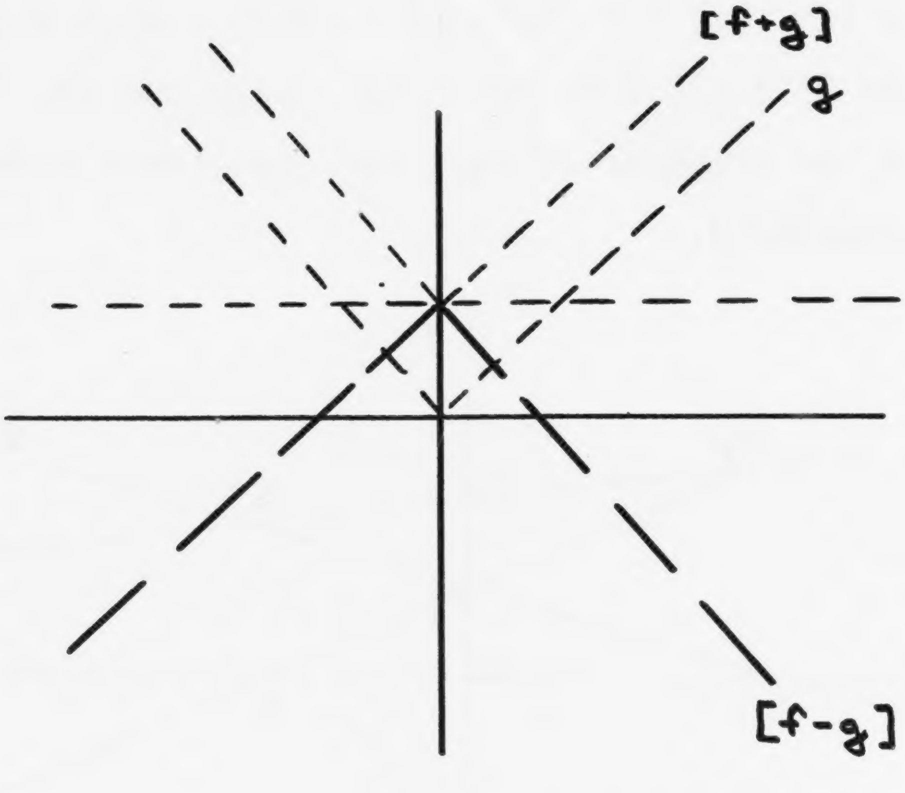
(ii) x $|x| + 3$ or $3 + |x|$

(iii) x $3 - |x|$

(iv) x $|x| - 3$

(v) x 3

(f)



2. (a)

x	h	k	[h + k]	[h - k]	[k - h]
0	0	-1	-1	1	-1
7	343	20	363	323	-323
12.5	1953.125	36.5	1989.625	1916.625	-1916.625
-14	-2744	-43	-2787	-2701	2701
-3	-27	-10	-37	-17	17

(b) (i) -1 (iv) 374

(ii) -1 (v) -343

(iii) 2744 (vi) -25

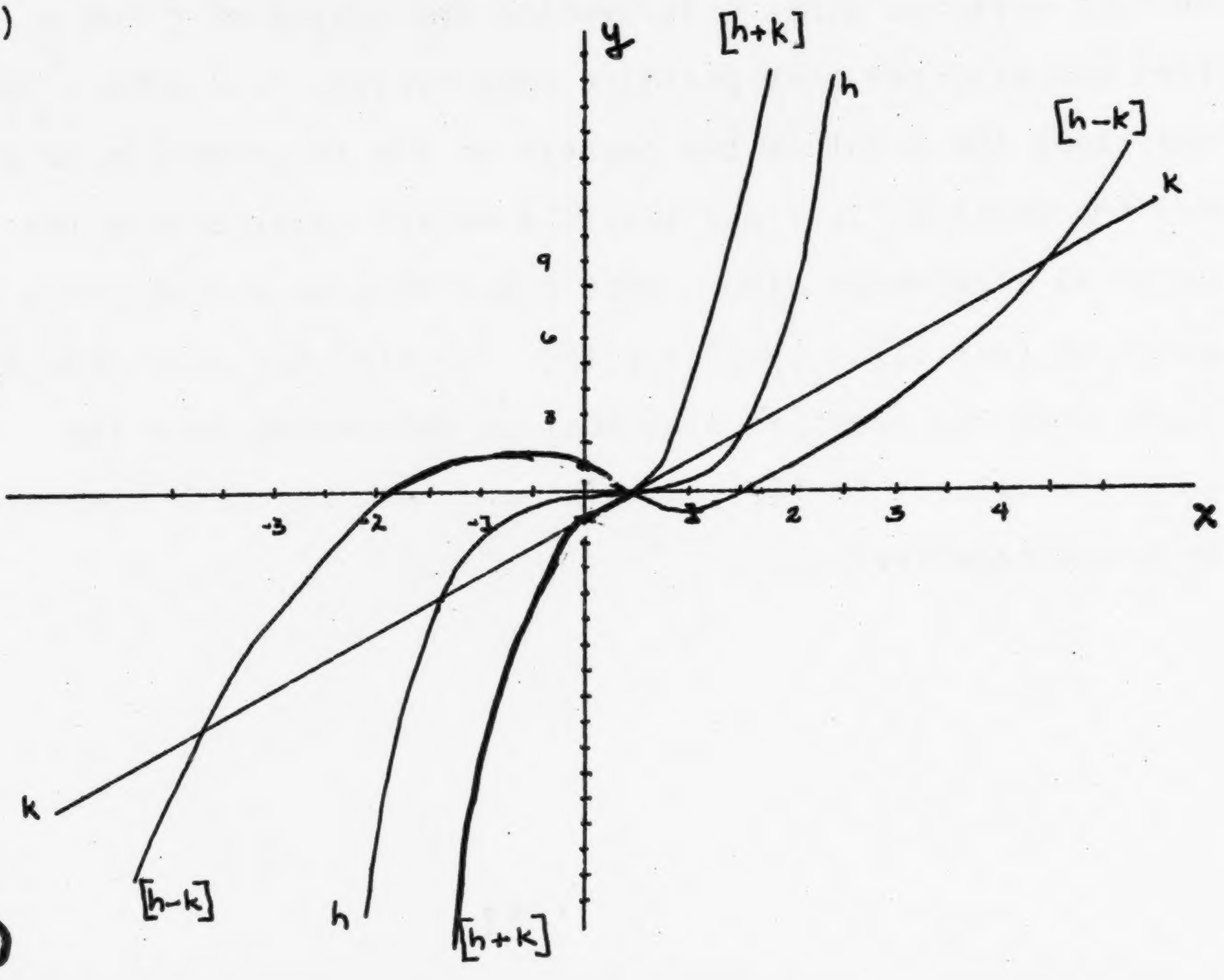
(c) (i) $x \xrightarrow{[h + k]} x^3 + (3x - 1)$

(ii) $x \xrightarrow{[h - k]} x^3 - (3x - 1)$

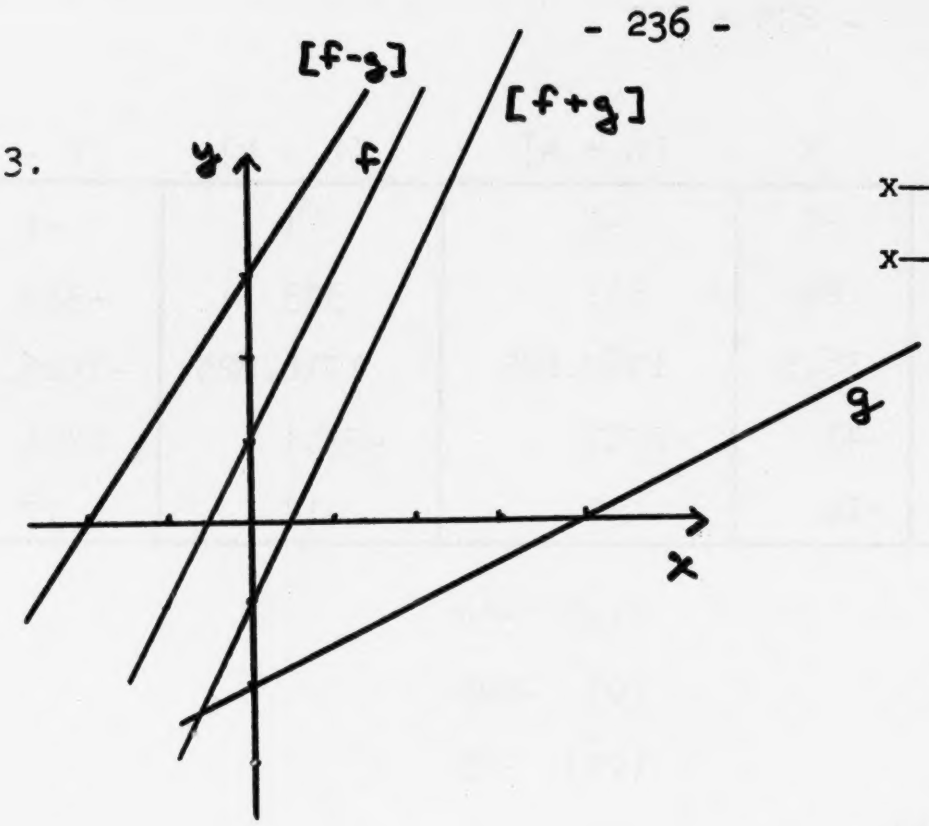
(iii) $x \xrightarrow{[k + h]} (3x - 1) + x^3$

(iv) $x \xrightarrow{[k - h]} (3x - 1) - x^3$

(d)



3.



$x \xrightarrow{f} 2x+1$

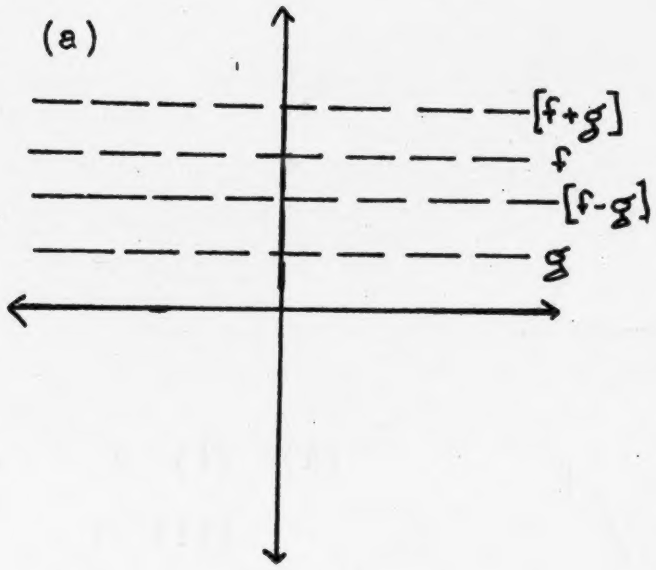
$x \xrightarrow{g} \frac{1}{2}x-2$

$x \xrightarrow{f+g} \frac{5}{2}x-1$

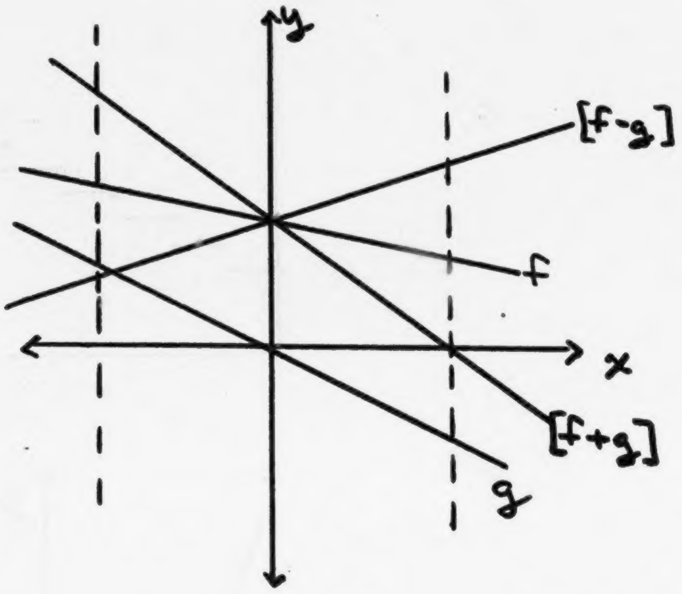
$x \xrightarrow{f-g} \frac{3}{2}x+3$

Note: In order to obtain the graph of [f + g](x) using the graph of f and g, use the following proceedure. Construct several vertical lines intersecting the graphs of f and g. If f(x) and g(x) are both positive then measure f(x) with a compass and place the point of the compass on the intersection of g(x) and the vertical line and inscribe an arc intersecting the vertical line above g(x). This point will be contained in the graph of [g + f](x) or [f + g](x). If f(x) and g(x) have different signs then the negative sign must be subtracted from the positive value. If f(x) and g(x) are both negative they must be added negatively.

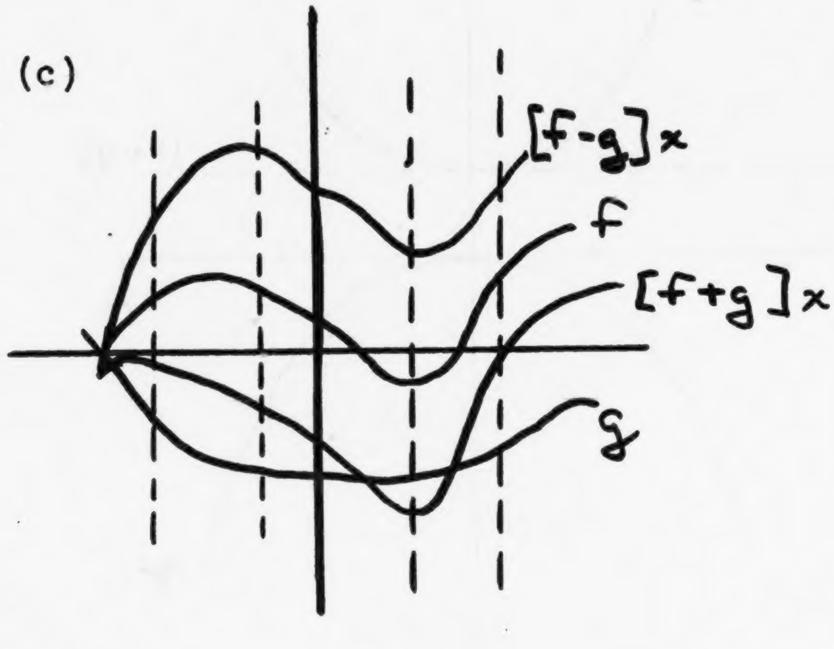
4. (a)



(b)

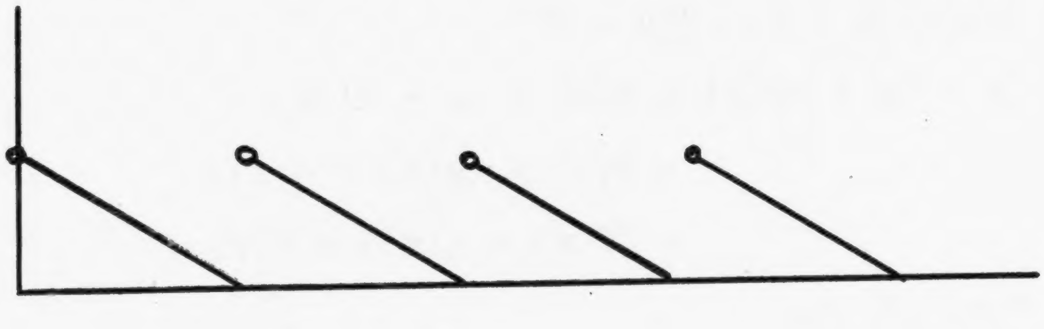


(c)

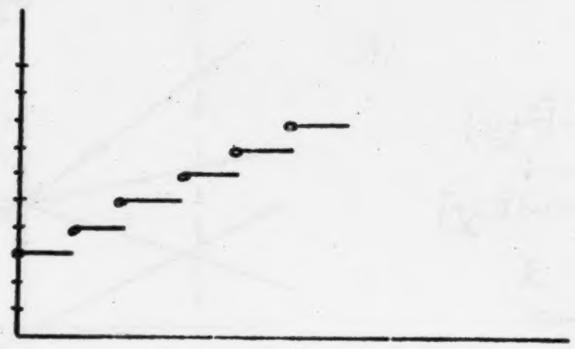


5.

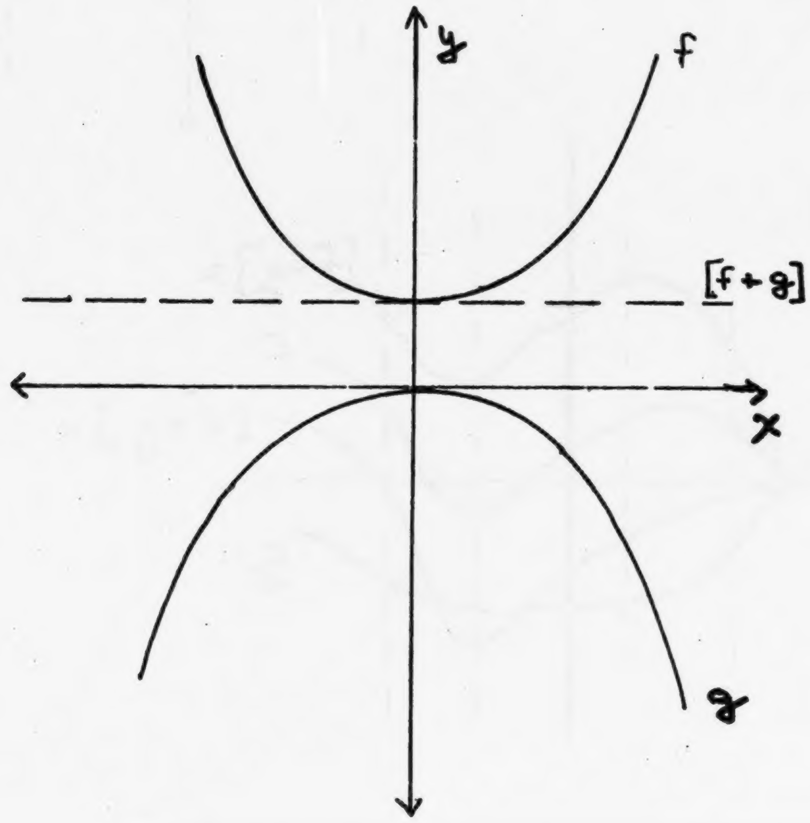
x	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$	3	$\frac{10}{3}$
p(x)	1	1	1	2	2	2	3	3	3	4
p(x) - X	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$



6.



7.



(a) (i) 1

(ii) 1

(iii) 1

(iv) 1

(v) 1

8. (a) (i) $(x + 2) + 3x$

(ii) $3x - x^2$

(iii) $[(x + 2) + 3x] - x^2$

(iv) $(x + 2) + (3x - x^2)$

(b) Yes.

(c) $$\begin{aligned}[f + [g + h](x) &= f(x) + [g + h](x)\\ &= f(x) + (g(x) + h(x))\\ &= (f(x) + g(x)) + h(x)\end{aligned}$$

(d) Yes, it is.

9. $x \xrightarrow{c} 0$ for all x.

10. (a) (i) 0
    (ii) 0
    (iii) 0
    (iv) 0

(b) $x \xrightarrow{[f + g]} 0$

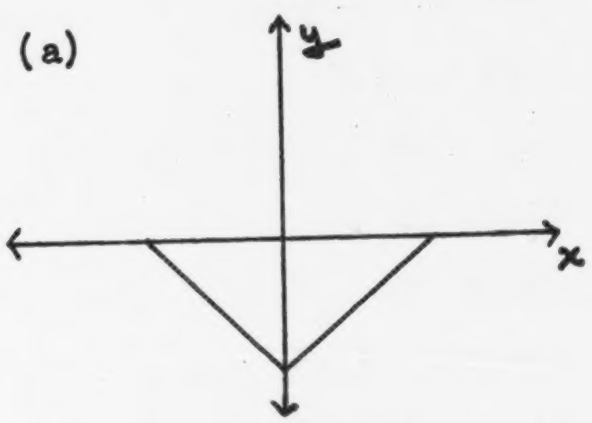
(c) Yes, because $h + [-h] = 0$

(d) $x \xrightarrow{h + [-h]} 0$

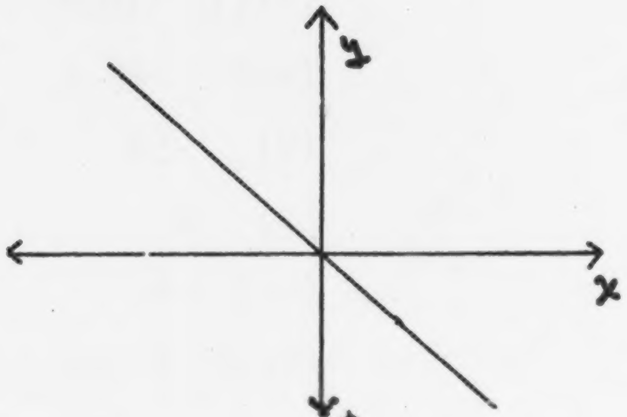
(e) Yes. Self-evident

(f) Yes, satisfies all group properties.

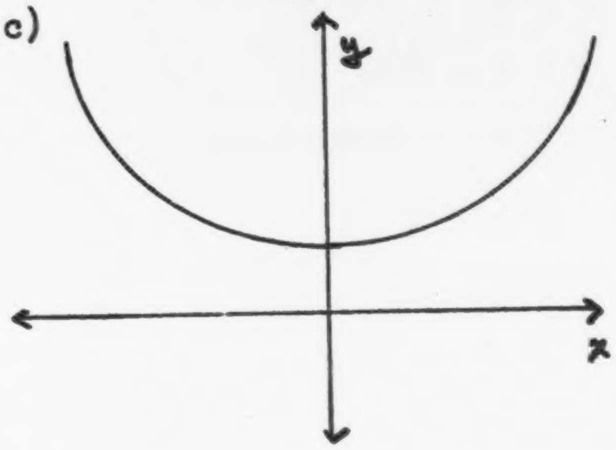
11. (a)

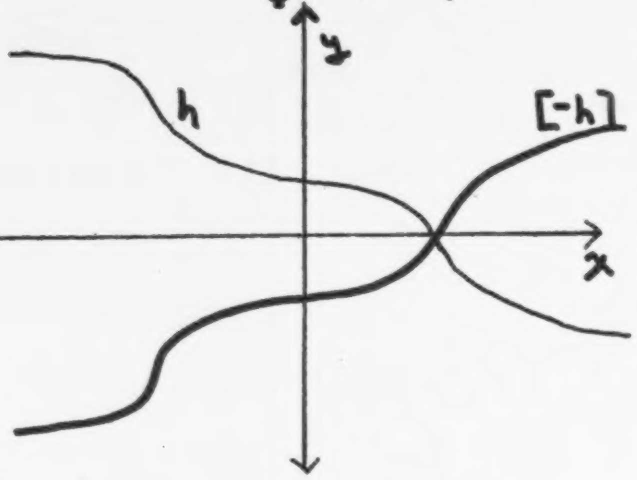


(b)



(c)





12. Yes. $[f + g](x) = f(x) + g(x) = g(x) + f(x) = [g + f](x)$

No. $[f - g](x) = f(x) - g(x) \neq g(x) - f(x) = [g - f](x)$

therefore, $[f - g](x) \neq [g - f](x)$

7.13 $[f - g]$ and $[\frac{f}{g}]$ (1 - $1\frac{1}{2}$ days)

The problem with defining $[\frac{f}{g}]$ for all f, g in F is the fact that $g(x)$ may be 0 for some x and thus $[\frac{f}{g}](x) = \frac{f(x)}{g(x)}$ cannot always be defined as an element of F. $[\frac{f}{g}]$ is a function with domain all x such that $g(x) \neq 0$.

7.14 Exercises

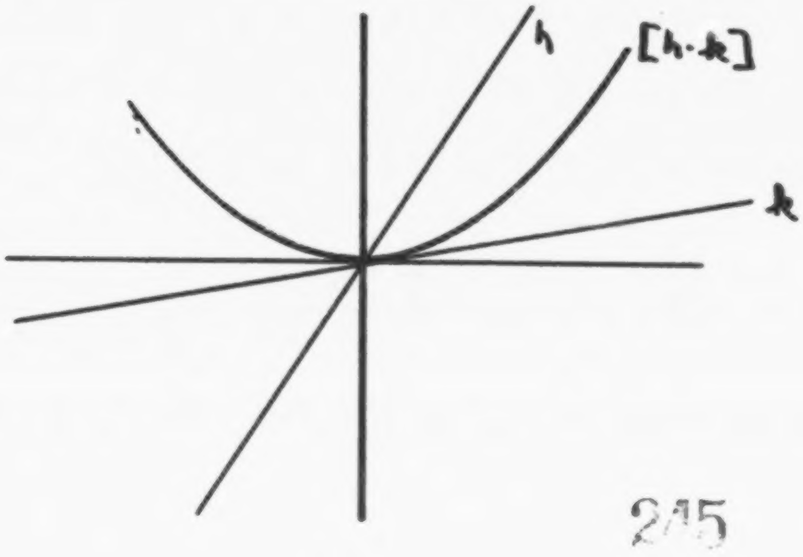
1. (a) (i) 0
(ii) 12
(iii) 16.8
(iv) -25.2
(v) -54
(vi) 0
(vii) 2
(viii) 2.8
(ix) -4.2
(x) -9

(b) (i) 0
(ii) 24
(iii) 47.04
(iv) 105.84
(v) 486
(vi) 0
(vii) 24
(viii) 47.04
(ix) 105.84
(x) 486

(c) (i) undefined
(ii) 6
(iii) 6
(iv) undefined
(v) $\frac{1}{6}$
(vi) $\frac{1}{6}$

(d) (i) 0
(ii) 4
(iii) 5.6
(iv) -18

2.



3.

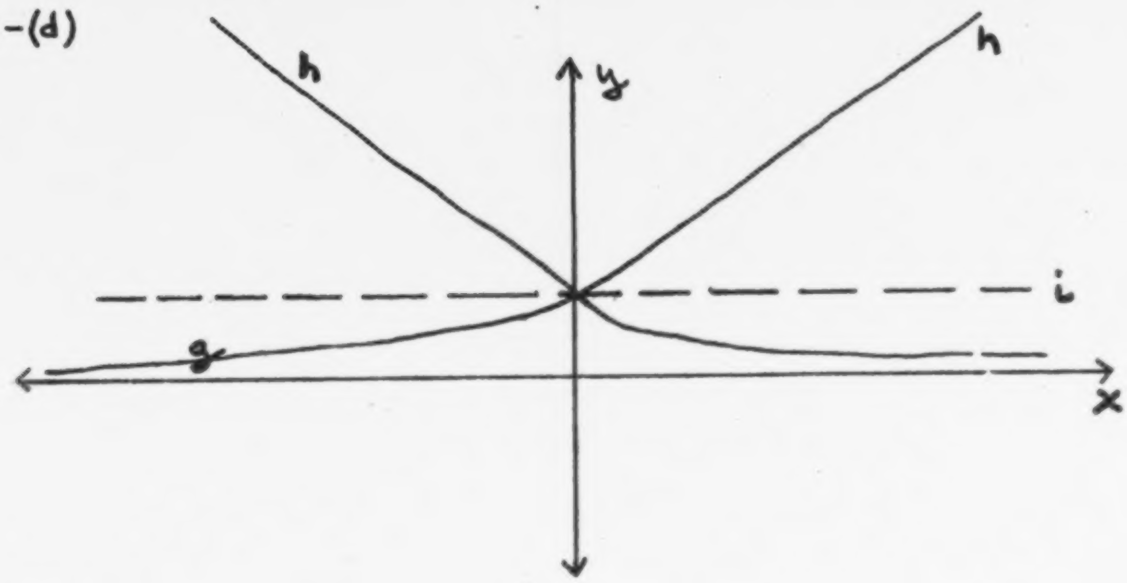
[f + g]	[f · g]
$12\frac{1}{2}$	$-6\frac{1}{2}$
$-20\frac{3}{4}$	-78
3	-340
1	0

4. (a) $\{y \in R: \ y \geq 1\}$, $\{y \in R: \ 0 < y \leq 1\}$, $\{i\}$

(b)

$\lvert x\rvert + 1$	1	2	2	3	3	4	4	$\frac{3}{2}$	$\frac{3}{2}$
$\frac{1}{\lvert x\rvert + 1}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{2}{3}$

(c)-(d)



5. (a) (i) 53 (iv) 1

(ii) 395 (v) 1

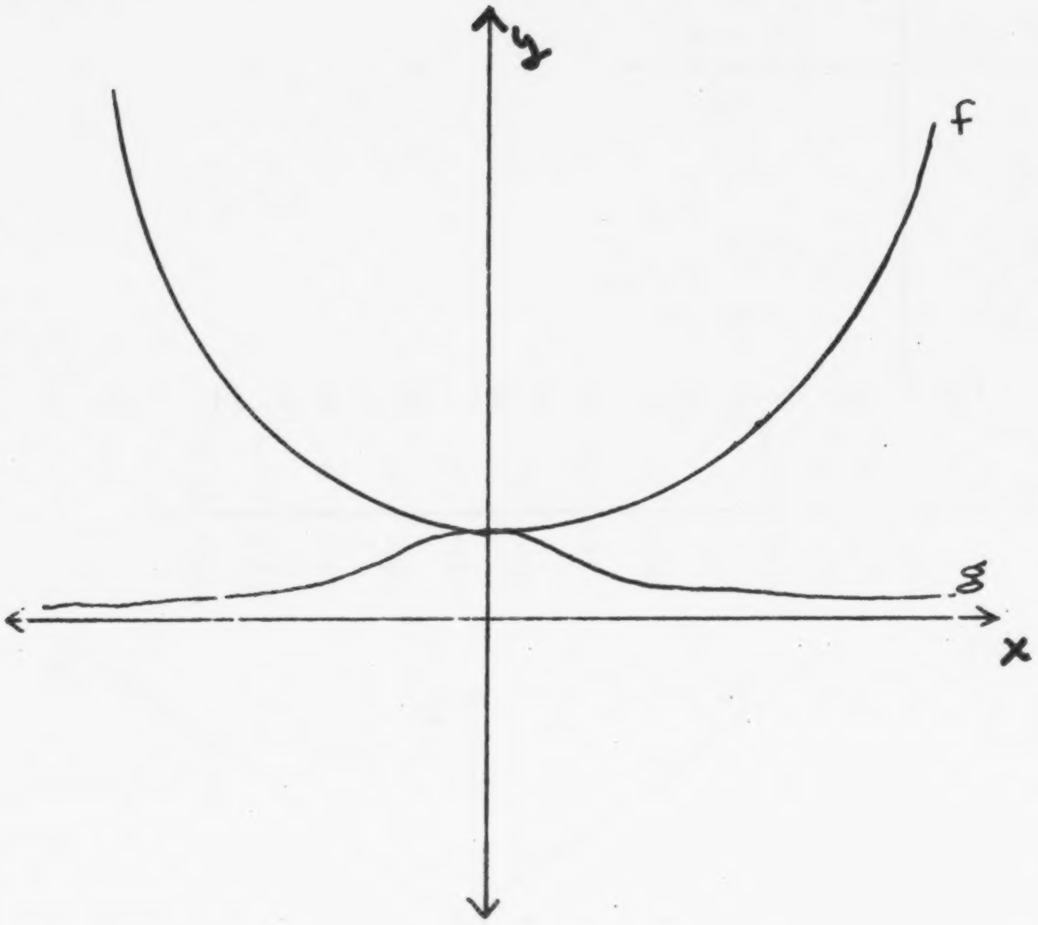
(iii) 87.5 (vi) 1

(b) (i) 53 (iv) 53

(ii) 395 (v) 395

(iii) 87.50 (vi) 87.5

6.



(e) (i) 1

(ii) $\frac{4}{5}$

(iii) $\frac{4}{5}$

(iv) $\frac{1}{2}$

(v) $\frac{1}{2}$

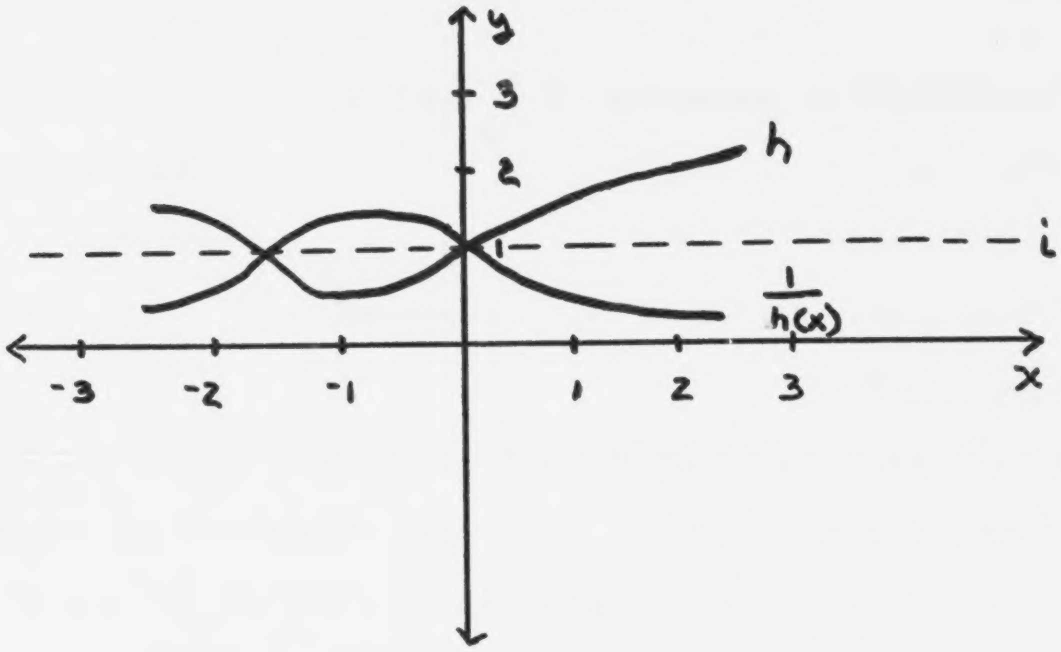
(vi) $\frac{4}{13}$

(vii) $\frac{4}{13}$

(viii) $\frac{1}{5}$

(ix) $\frac{1}{5}$

7.



Note: It must be noted that this function may not be defined for all values of x. The above graph shows approximate values of the function i.e.:

$$h(x) = \frac{1}{h(x)} = 1$$
$$h(x) = \frac{3}{2} \; ; \; \frac{1}{h(x)} = \frac{2}{3}$$
$$h(x) = 2 \; ; \; \frac{1}{h(x)} = \frac{1}{2}$$

8. (a) $[f \cdot [g \cdot h]](x) = f(x) \cdot [g \cdot h](x)$

$= f(x) \cdot (g(x) \cdot h(x))$

$= (f(x) \cdot g(x)) \cdot h(x)$

(b) · is associative.

9.

x	f	g	h	[f · g]	[f · h]	[f · g] + [f · h]
2	8	5	4	40	32	72
0	0	-1	0	0	0	0
-2	-8	-7	-4	56	32	88

10.

[g + h]	f · [g + h]
9	72
-1	0
-11	88

11. Distributive property of · over +.

12. Commutative + · Distributive
Associative + · Inverses +
Identities $x \longrightarrow 0$, $x \longrightarrow 1$
Cancellation +

After this exercise a good class discussion might arise from comparing (F, +, ·) to the structure of number systems. It might also be interesting to examine (F, +, 0) and (F, ·, 0). Let the students experiment and arrive at conjectures of properties--no fixed list need be developed at this stage.

13. (a) (i) 9, (ii) 39, (iii) -33.
(b) $x \xrightarrow{3f} (6x + 9)$
(c) (i) 0 (ii) 75 (iii) 147
(d) $x \xrightarrow{3g} 3x^2$
(e) (i) 9
(ii) 114
(iii) 114
(iv) 9
(v) 114
(vi) 114

The operation introduced here is scalar multiplication in the group of real functions (F, +). The inclusion of this new external operation makes the set of functions a vector space, but this is for the future.

7.15 The Square Root and Cube Root Functions (2 - 3 days)

This section looks at square and cube root assignment as a process defined at every point of the line (positive x for √), not as the random assignments √2, √5, √7, √4 etc. which are usually generated by equation solution. The graphs of $\sqrt{}$ and $\sqrt[3]{}$ are generated using symmetries in $y = x$.

7.16 Exercises

1. (a) 2.2
    (b) 2.4
    (c) 2.6 (answers will vary)
    (d) 1.6
    (e) 2.1

2. Answers will vary

3. (a) 2
    (b) 1.8
    (c) 1.9
    (d) 1.6
    (e) 1.3

4. Answers will vary.

5. (a) $x < 0$ or $x > 1$
    (b) $0 < x < 1$
    (c) $x = 0$, $x = 1$
    (d) $x = 0$, $x = 1$
    (e) $x>1$
    (f) $0 < x < 1$
    (g) $0< x < 1$
    (h) $x > 1$
    (i) $x = 0, 1$
    (j) $-1 < x < 0$ or $x > 1$
    (k) $x < -1$ or $0 < x < 1$
    (l) $x = \pm 1, 0$
    (m) $x = \pm 1, 0$
    (n) $-1 < x < 0$ or $x > 1$
    (o) $x < -1$ or $0 < x < 1$
    (p) $x < -1$ or $0 < x < 1$
    (q) $-1 < x < 0$ or $x > 1$
    (r) $x = 0, \pm 1$

6. (a) 11

(b) 10

(c) 8

(d) 4

(e) $2\sqrt{11}$

(f) $7\sqrt{3}$

(g) $10\sqrt{5}$

(h) $-3\sqrt[3]{4}$

(i) $\frac{3}{2}\sqrt{2}$

(j) $\frac{1}{2}$

(k) $\frac{1}{5}\sqrt{15}$

(l) $\frac{1}{4}\sqrt{2}$

(m) $-\frac{1}{2}\sqrt[3]{6}$

(n) $\frac{5}{2}\sqrt{2}$

(o) $\frac{1}{5}\sqrt{15}$

(p) $-\frac{1}{3}\sqrt[3]{6}$

7. (a) $7\sqrt{2}$

(b) -3

(c) $-\frac{1}{9}\sqrt{6}$

(d) $\frac{15}{2}\sqrt{2}$

(e) 90

(f) 4

(g) $10 + 5\sqrt{3} - 2\sqrt{2} - \sqrt{6}$

(h) $2\sqrt{15} - 5\sqrt{10} + 4\sqrt{6} - 20$

(i) -1

(j) 26

8. $g(x) = x^3$

$g(ab) = (ab)^3 = (ab)(ab)(ab) = (aaa)(bbb) = a^3b^3 = g(a) \cdot g(b)$.

$\therefore$ f is multiplicative.

$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$ iff $(\sqrt[3]{ab})^3 = (\sqrt[3]{a} \cdot \sqrt[3]{b})^3$, by definition of $\sqrt[3]{}$.

But,

$(\sqrt[3]{ab})^3 = ab$ by definition of $\sqrt[3]{}$.

And,

$(\sqrt[3]{a} \cdot \sqrt[3]{b})^3 = (\sqrt[3]{a})^3(\sqrt[3]{b})^3$ by multiplicative property of g.

Now, $(\sqrt[3]{a})^3 = a$ and $(\sqrt[3]{b})^3 = b$.

$\therefore$

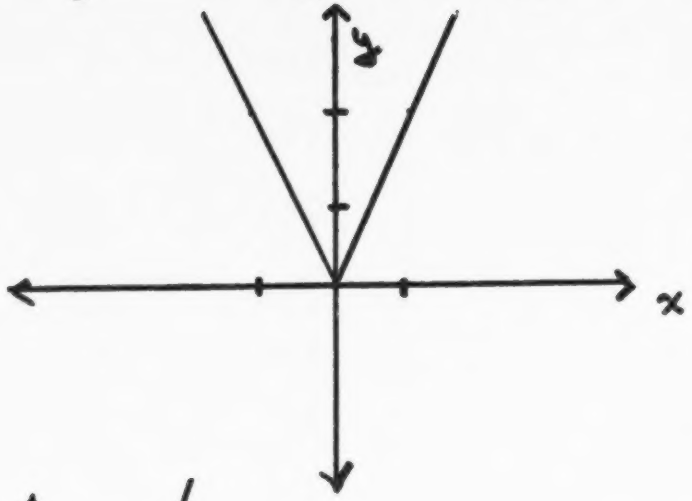
$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b}$.

7.18 Review Exercises

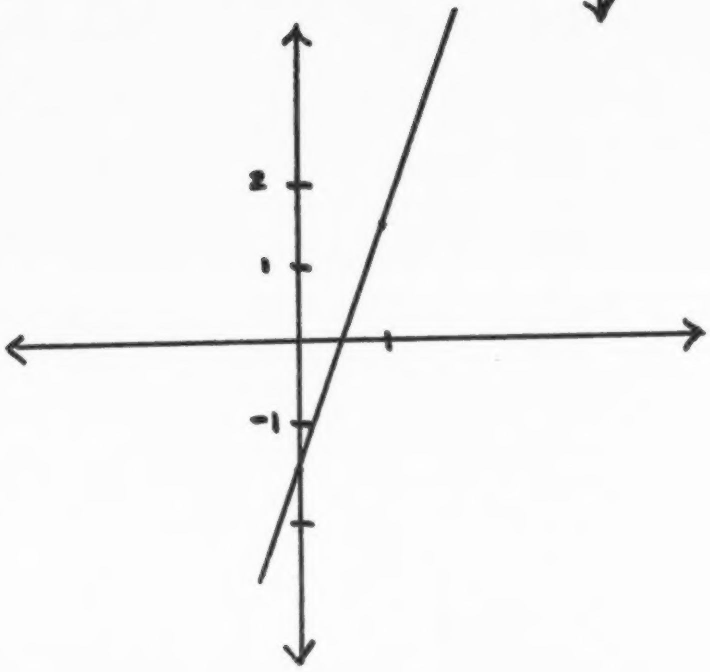
1.

x	h	g	goh	[h + g]	[h · g]
0	0	-1	-1	-1	0
1	1	1	1	2	1
-1	1	-3	1	-2	-3
$2\frac{1}{2}$	$\frac{25}{4}$	4	$\frac{23}{2}$	$\frac{41}{4}$	25
$-6\frac{1}{3}$	$\frac{361}{9}$	$\frac{-41}{3}$	$\frac{713}{9}$	$\frac{238}{9}$	$\frac{14801}{27}$
12	144	23	287	167	3312
-19	361	-39	721	322	14079
$\sqrt{2}$	2	$2\sqrt{2} - 1$	3	$2\sqrt{2} + 1$	$4\sqrt{2} - 2$
$\|-3\|$	9	5	17	14	45

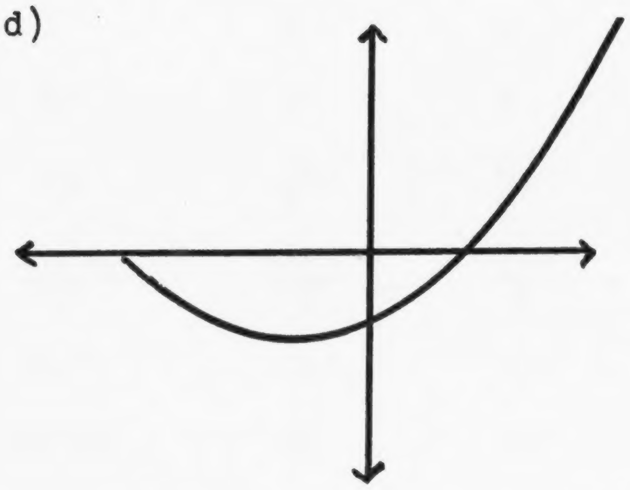
2.

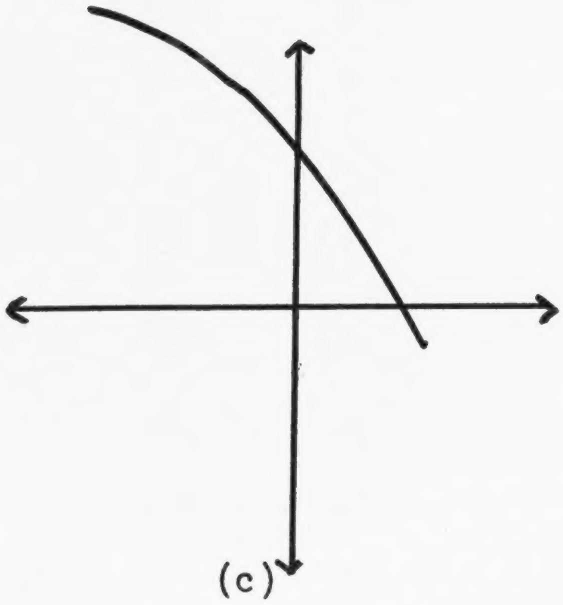


3.



4. (a) no yes
(b) 0 -1
(c) 3 1
(d)





5.

	(a)	(b)
(1)	No, $\lvert\pm 1\rvert = 1$	No, $? \longrightarrow -2$
(2)	Yes	Yes
(3)	Yes	Yes
(4)	Yes	Yes
(5)	No, ±1 3.14	No, $? \longrightarrow 2$
(6)	Yes	Yes

(c)

$x \quad g^{-1} \quad \sqrt[3]{x}$

$x \quad h^{-1} \quad \frac{1}{17}x + 17$

$x \quad k^{-1} \quad x - \sqrt{2}$

$x \xrightarrow{n^{-1}} \frac{1}{5}x$

6. (a) 7.51
(b) -2197
(c) 0
(d) 0
(e) 3.14
(f) 1.25
(g) 3.14
(h) 15.7
(i) -153
(j) -425
(k) 3390
(l) -3360
(m) 50,000
(n) 50,000
(o) -340
(p) -81

7. (a) $x \xrightarrow{[h + n]} 22x - 289$

(b) $x \xrightarrow{[h \cdot n]} 85x^2 - 1445x$

(c) $x \xrightarrow{[n \circ m]} 15.7$

(d) $x \xrightarrow{[m \circ n]} 3.14$

(e) $x \xrightarrow{n \cdot g]} 5x^4$

(f) $x \xrightarrow{[g \circ n]} 125x^3$

(g) $x \xrightarrow{[5g]} 5x^3$

(h) $x \longrightarrow (x + \sqrt{2})^3$ or

$x \longrightarrow x^3 + 3x^2\sqrt{2} + 6x + 2\sqrt{2}$

Suggestions for test items for Chapter 7

I. Let the following functions from $\mathbb{R}$ to $\mathbb{R}$ be as follows:

$f: x \longrightarrow |x| + 2$

$g: x \longrightarrow x^2$

$h: x \longrightarrow 5x - 3$

Complete the following table:

x	f	g	h	f o g	h + f	h · g	$\frac{h}{g}$	g o f	g - f	4h
$\frac{5}{4}$										
$-6\frac{1}{3}$										

II. Graph the following functions on one coordinate axes:

(a) $\ell: x \longrightarrow \frac{5}{2}x - 2$

(b) $m: x \longrightarrow 2x^2 - 5$

(c) $n:\ x \longrightarrow 2|x| + 3$

(d) $[\ell + (m+n)](x)$ Hint: Use addition of ordinates

III. (a) Graph the following functions on one coordinate axes:

$y = x + 2$, $y = x^3$, $y = \sqrt[3]{x}$

(b) From the above graphs, estimate the solution of the following to the nearest tenth:

1) $\sqrt[3]{x} = x + 2$

2) $\sqrt[3]{x} > x + 2$

3) $\sqrt[3]{x} < x + 2$

4) $x^3 = x + 2$

5) $x^3 > x + 2$

6) $x^3 < x + 2$

7) $\sqrt[3]{x} > x^3$

IV. For the graph of f given below -

a) Give the rule for f.

b) Is this function one-to-one? Explain why or why not.

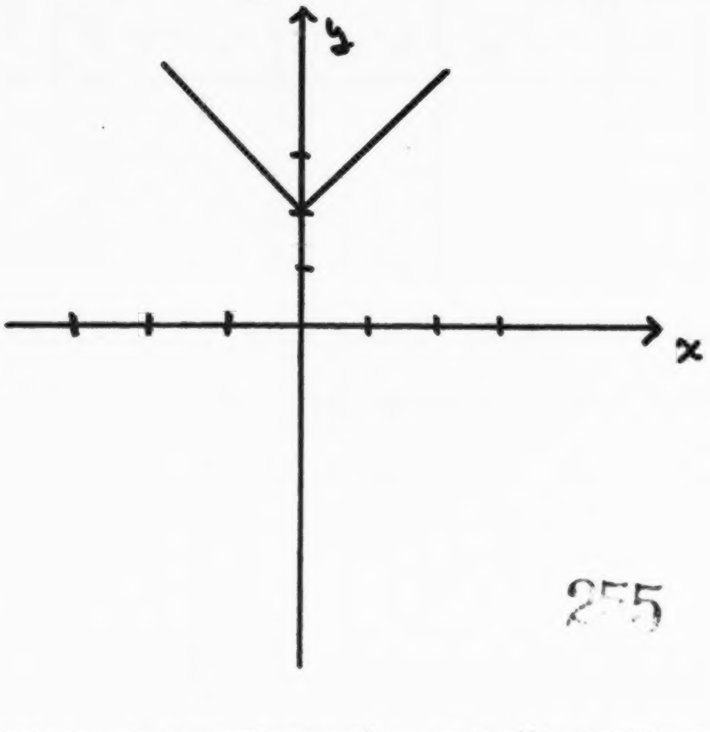
c) Is this function onto? Explain why or why not.

d) What is the image of 3?

e) What is the pre-image of 5?

f) Sketch the graph of -f on the same coordinate axes.

g) Sketch the graph of $\frac{1}{f}$ on the same coordinate axes.

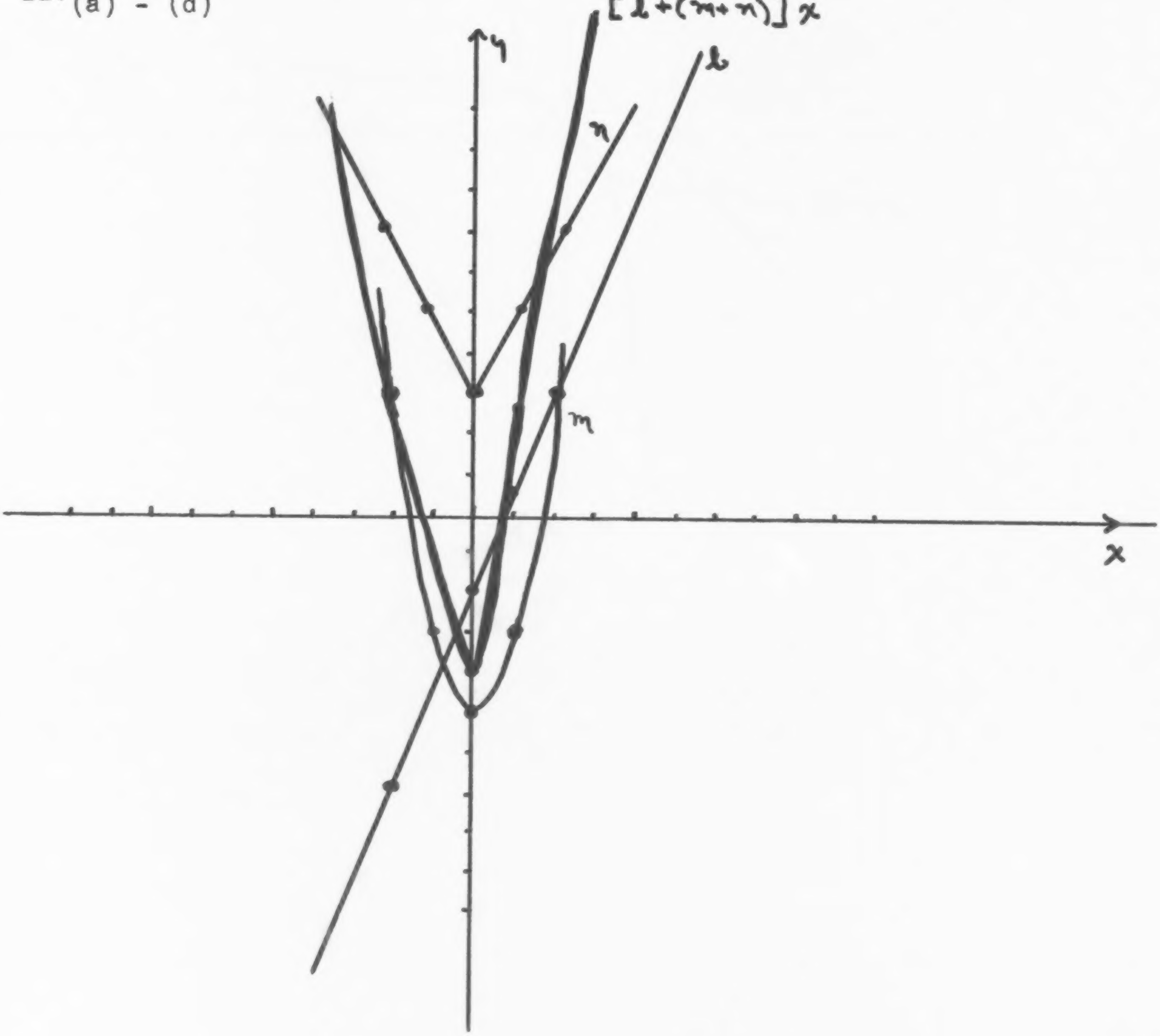


Solution to chapter test

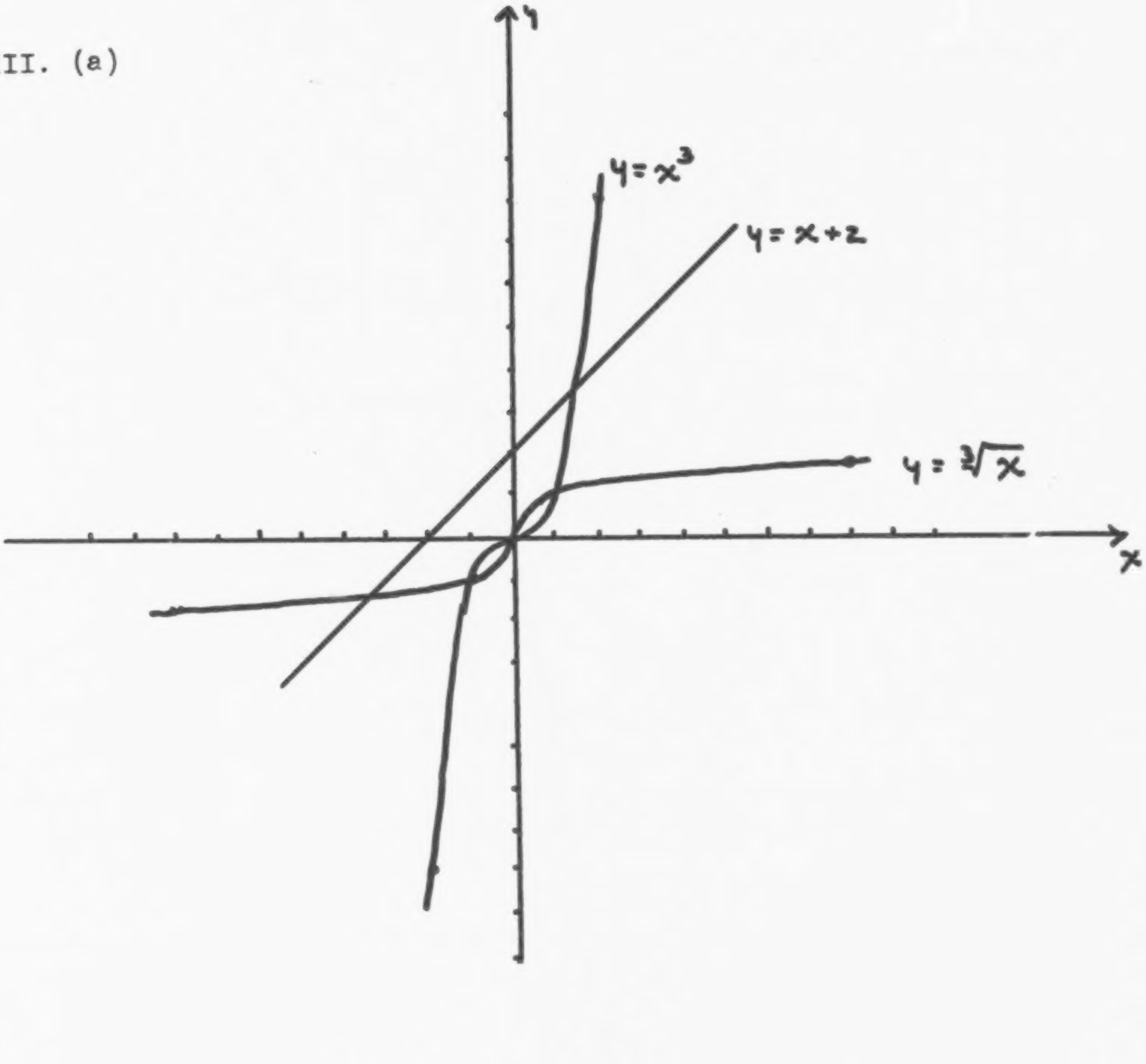
I.

x	f	g	h	$f \circ g$	$h + f$	$h \cdot g$	$\frac{h}{f}$	$g \circ f$	$g - f$	$4h$
$\frac{5}{4}$	$\frac{13}{4}$	$\frac{25}{16}$	$\frac{13}{4}$	$\frac{57}{16}$	$\frac{13}{2}$	$\frac{325}{64}$	1	$\frac{169}{16}$	$\frac{-27}{16}$	13
$-6\frac{1}{3}$	$\frac{25}{3}$	$\frac{361}{9}$	$\frac{-104}{3}$	$\frac{379}{3}$	$\frac{-79}{3}$	$\frac{-37{,}544}{27}$	$\frac{-104}{25}$	$\frac{625}{9}$	$\frac{286}{9}$	$\frac{-416}{3}$

II. (a) - (d)



III. (a)



(b)

1) ≈ -3.6

2) $x < \approx -3.6$

3) $x > \approx -3.6$

4) ≈ 1.5

5) $y > \approx 1.5$

6) $x < \approx 1.5$

7 $x < -1$ or $0 < x < 1$

IV. a) $f: x \longrightarrow |x| + 2$

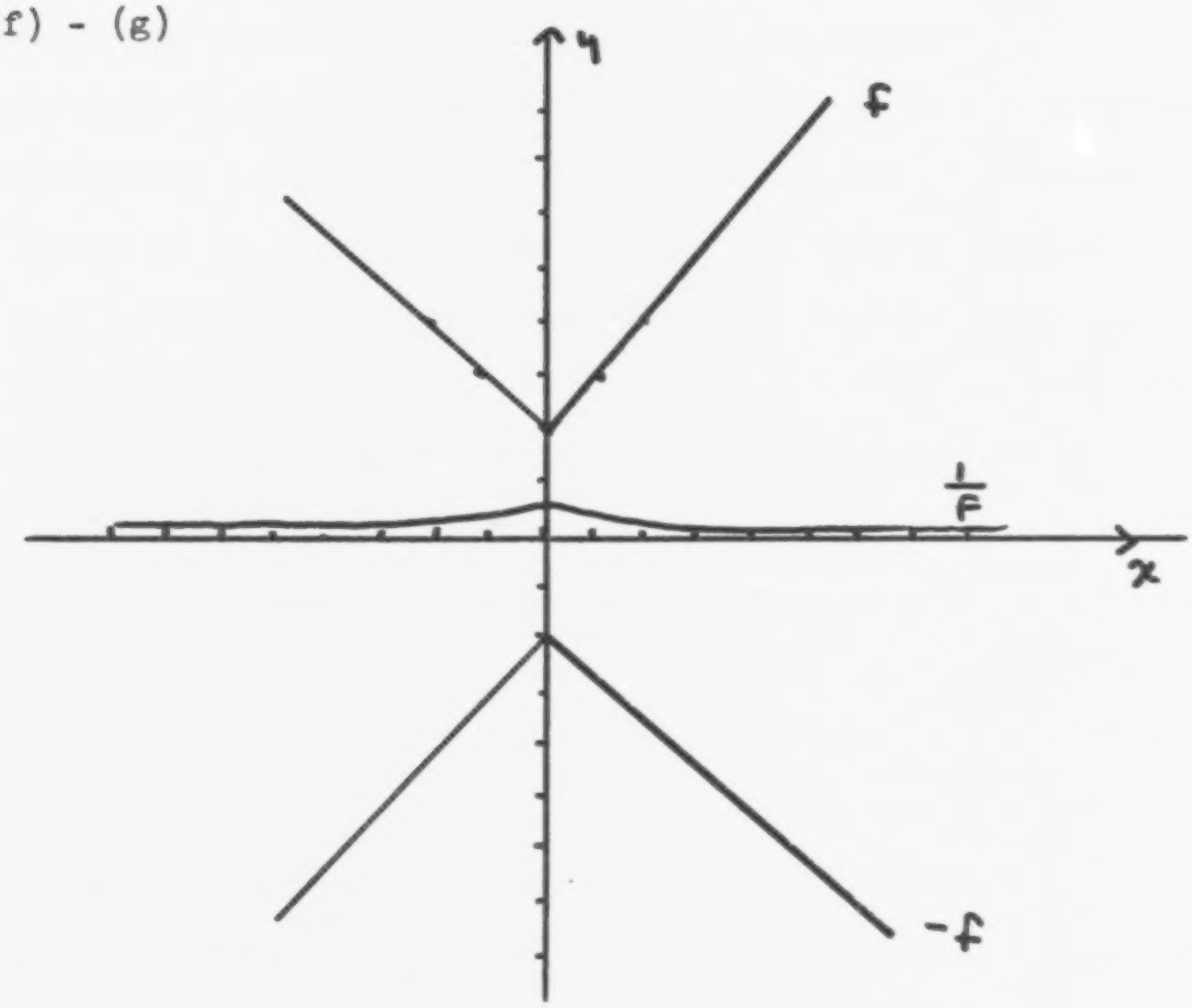
b) No - one image has two pre-images.

c) No - codomain ≠ range

d) 5

e) ±3

(f) - (g)



Chapter 8

Descriptive Statistics

Time estimate for chapter: 14 days with review

Introduction (Time estimate for 8.1, 8.2, 8.3 = 2 days)

The purpose of this chapter is to review and extend the material begun in Chapter 5 of Course I--with special emphasis on techniques for collecting statistical data, summarizing it by means of graphs and tables, and analyzing it by means of measures of central tendency and dispersion, such as the mean and variance.

The concepts of probability which were introduced in Chapter 5 of Course I will be picked up and studied further in Course III. However, especially in connection with the exercises in 8.3, the ideas of an experiment, a sample, a random sample, sample space, probability, etc. can profitably be mentioned and noted.

The experiments and problems of Section 8.3 (as well as the illustrative example of Section 8.2) are key problems in the sense that they are referred to and continued throughout the chapter. For that reason it is important that students retain copies of their original data--and whatever they do with the data--throughout this chapter. To aid teachers in choosing problems to be assigned, a list of problems dependent on others will be found at the end of this commentary.

Though some of the computation called for--especially in the latter parts of the chapter--may seem time consuming and tedious, it provides an excellent opportunity to review and drill on important arithmetical computational skills. It is worthwhile, for example, to compute means and variances by both the long method and by the short-cuts of Section 8.10.

8.2 Examples of Sets of Data and Their Graphical Presentation

The graphic techniques of this chapter are the frequency diagram, the cumulative frequency diagram, the frequency histogram, and the frequency and cumulative frequency polygons. It is important that students be able to relate these to each other and to the data tables from which they are obtained. One possible technique might be to start with one set of data, for example that of Table 8.1, and construct in succession a frequency diagram, a cumulative frequency diagram, a frequency histogram, and finally, frequency and cumulative frequency polygons. (Note that the frequency polygon may be drawn directly on the graph of the frequency histogram, as shown in Figure 8.5). While it would be possible to construct a cumulative frequency histogram, it is less frequently used than the others and is not discussed in this chapter.

One useful property of the frequency histogram is that since each bar in the histogram has the frequency for its height and the measure for its base, the total area of the histogram for a distribution is equal to the total number of observations. This might be pointed out in passing for the benefit of the better students.

The definitions of *range*, *median*, *mode* and *quartile* should be discussed and understood. Before computing the *median* the data must be ordered. Also, be sure to stress the fact that the frequency must be taken into consideration.

Example: The set {1,1,1,4,3} has median = 1, *not* 3 as does {1,3,4}

In definition 2, the phrase "the middle measure" may be misunderstood without this extra discussion.

Note that a distribution may have more than one mode.

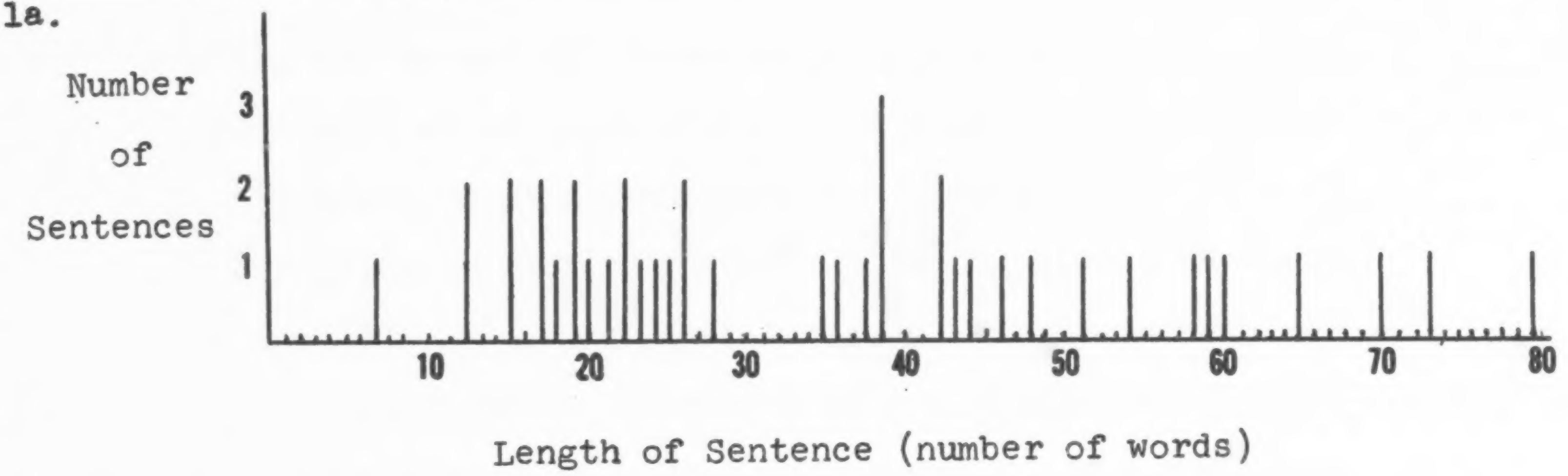
Note also that the xth percentile is that point on the cumulative frequency graph at or below which there are x percent of the measurements. For example, in Figure 8.6, there were 9 bulbs (or 18%) which had a life of less than or equal to 975 hours, so 975 is the 18th percentile.

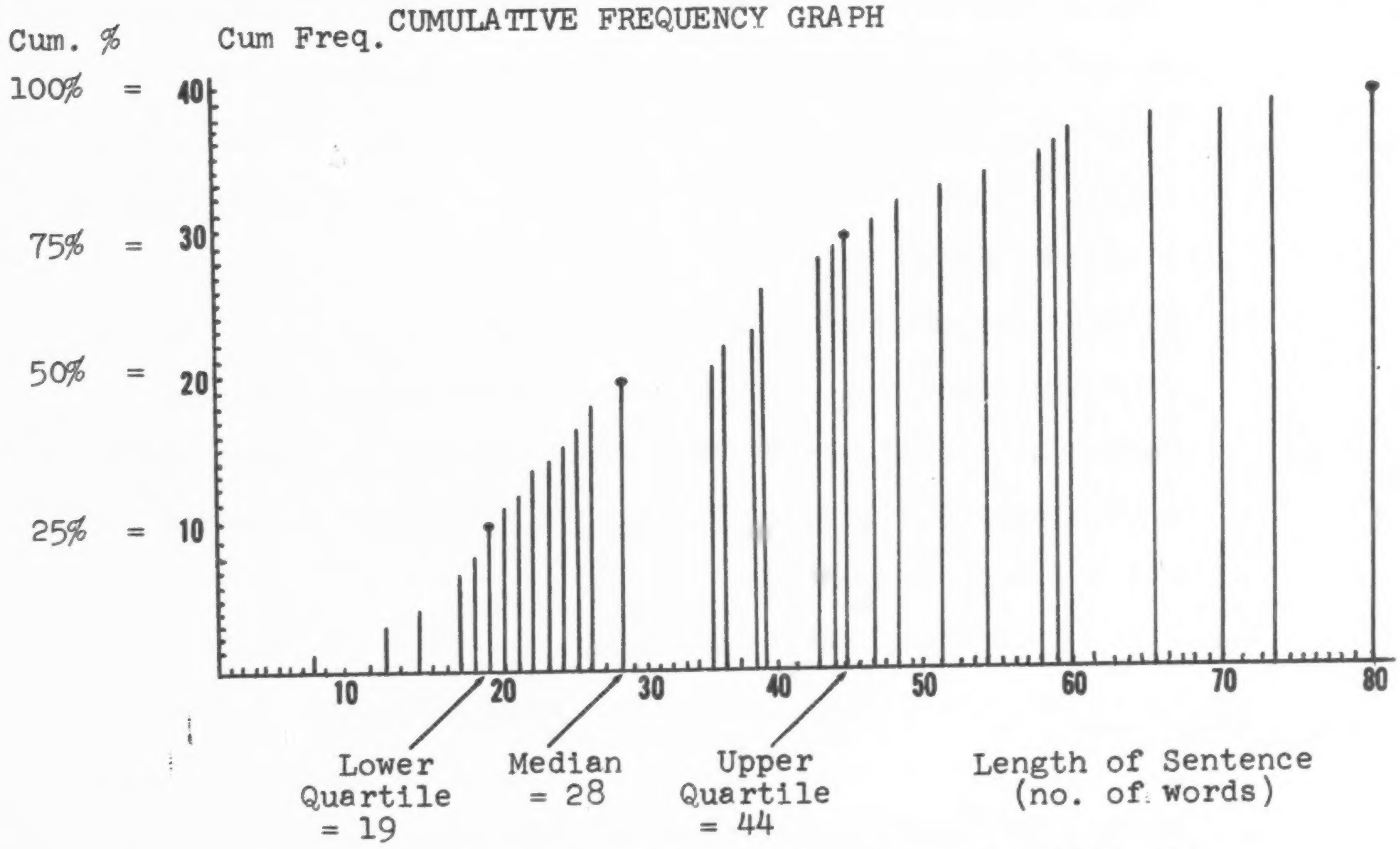
8.3 *Exercises*

Students should not be expected to do all the exercises in this section. At least two of Problems 1-5 should be assigned, including either Problem 3 or Problem 4, either Problem 6 or 8, and at least one of 9-11. In this section as in succeeding ones, how many and which problems are assigned must to a large degree depend on the teacher's judgment of the students' comprehension of the material. It may be desirable to assign some of the exercises in this section later in the chapter.

Solutions to Problems

1a.





Range: 73; Median: 28; Lower Quartile: 19; Upper Quartile: 44

2. To estimate the mean: Each card has an equally likely probability of being drawn. Hence the mean (expectation) for a draw of one card would be $\frac{1}{13}(1 + 2 + \ldots + 13) = \frac{1}{13}\left(\frac{13 \cdot 14}{2}\right) = 7$. For three cards drawn, with replacement, the expected value for the sum would be $7 + 7 + 7 = 21$. We may use this as a good estimate for the median without replacement, assuming the distribution to be symmetric. (This exercise might be a good one to do in class. Students should actually obtain the mean by doing the experiment; the above calculation is for the teacher's benefit and should not be presented to the class.)
3. Since the probability of beads is 1/2, the expected sample mean is $1/2 \cdot 20$ or 10, so the example median may be estimated to be 10.
4. Since the maximum sum is 18 and the minimum sum is 8, the expected sample mean is 10 1/2, so the sample median may be estimated to be 10 1/2.
5. We would expect each digit of the set $\{0, 1, \ldots 9\}$ to be equally likely (i. e., to occur approximately the same number of times). 5(c) asks for summarizing. Have the students make a frequency table, a frequency histogram and a frequency polygon.
6. Part of the purpose of this problem, and of Problems 9-11, is to let the students construct a table completely on their own, with no information (other than the data) being given. At least one problem of this type should be

assigned as a home exercise.

7-8. Answers are in the text. (Figure 8.6).

9-11. Answers will vary. Students' solutions should be compared and discussed.

8.4 The Symbol $\sum$ and Summation (Time estimate for 8.4 and 8.5 = 2 days)

This section will give students their first formal introduction to summation and use of the symbol $\sum$. Since "$\sum$" is used extensively in the rest of this chapter and is essential for much later work in mathematics, it is important that this section be covered carefully. However, the teacher should keep in mind that this is only a first introduction, and should not expect all students to exhibit perfect comprehension. This will come with practice and review in this and later courses.

Some enrichment material could be introduced here. For example, what is the meaning of $\sum_{i=1}^{5} x_{2i}$ or $\sum_{i=3}^{7} x_{i-2}$?

8.5 Exercises

All parts of Problems 1 and 2 should be assigned and gone over in class. Problems 3-7 involve derivation of various properties in summation. Since these properties are used in later work, the results of these exercises should be stressed. Every student should have a good understanding of how the results of Problems 3, 4, and 6 are obtained, and in particular how these results depend on properties of the real numbers (commutativity, associativity, etc.). Problem 7 is a combination of the results of the preceding exercises. While not all students will be able to do this problem unaided, it should be assigned (perhaps on a "try it" basis) and gone over in class.

Answers to Exercises

1. a. 15 b. 71 c. 75 d. 35 e. $15 + 4k$

2. a. 28 b. 70 c. 30

3. $$\begin{aligned}\sum_{i=1}^{n} kx_i &= (kx_1 + kx_2 + kx_3 + \ldots + kx_n)\\ &= k(x_1 + x_2 + x_3 + \ldots + x_n)\\ &= k\sum_{i=1}^{n} x_i\end{aligned}$$

4. $$\begin{aligned}\sum_{i=1}^{n}(a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \ldots + (a_n + b_n)\\ &= (a_1 + a_2 + a_3 + \ldots + a_n) + (b_1 + b_2 + b_3 + \ldots b_n)\\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\end{aligned}$$

   a. If $n = 3$, we have

$$\begin{aligned}\sum_{i=1}^{3}(a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3)\\ &= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)\\ &= \sum_{i=1}^{3} a_i + \sum_{i=1}^{3} b_i\end{aligned}$$

b. The summation of the sum of two sets of terms is equal to the sum of their separate summations.

5. $$\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n}[a_i + (-b_i)]$$

$$= \sum_{i=1}^{n}a_i + \sum_{i=1}^{n}(-b_i) \qquad \text{(by Exercise 4)}$$

$$= \sum_{i=1}^{n}a_n + (- \sum_{i=1}^{n}b_i) \qquad \text{(by Exercise 3)}$$

$$= \sum_{i=1}^{n}a_n - \sum_{i=1}^{n}b_i \qquad \text{(def. of Subt.)}$$

6. $$\sum_{i=1}^{n}k = \underbrace{k + k + k + k + \ldots + k}_{n \text{ terms}} = nk$$

7. $$\sum_{i=1}^{n}(x_i - m)^2 = \sum_{i=1}^{n}(x_i^2 - 2x_i m + m^2)$$

$$= \sum_{i=1}^{n}x_i^2 - \sum_{i=1}^{n}2x_i m + \sum_{i=1}^{n}m^2 \qquad \text{(by Exercise 4 and 5)}$$

$$= \sum_{i=1}^{n}x_i^2 - 2m\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}m^2 \qquad \text{(by Exercise 3)}$$

$$= \sum_{i=1}^{n}x_i^2 - 2m\sum_{i=1}^{n}x_i + nm^2 \qquad \text{(result of Exercise 6)}$$

8.6 The Arithmetic Mean, Its Computation and Properties Time estimate including 8.6 and 8.7 = 3 days)

In this section the mean is introduced, using the summation symbolism developed in Section 8.4. Students may question the necessity of this new statistic, since they are already familiar with the median and the mode. It might be desirable to indicate (without attempting to give reasons) that the mean is mathematically more useful than either the median or the mode. In fact, work from this point on will (of these three statistics) be mostly concerned with the mean.

Students should realize that the formula for the mean using frequencies of measurements is merely a computational device that comes directly from the basic definition. One way of introducing this might be to start with several measurements of one value and one or two having other values. For example, suppose $x_1 = 3$, $x_2 = 3$, $x_3 = 3$, $x_4 = 3$, $x_5 = 8$. Then from the definition we can write

$$\bar{x} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5} = \frac{3 + 3 + 3 + 3 + 8}{5} .$$

Students will see immediately that we could write

$$\bar{x} = \frac{4 \cdot 3 + 8}{5} .$$

4 is simply the number of measurements having value 3, or the frequency of that measurement.

It should be emphasized that the addition and multiplication properties of the mean discussed in this section are direct consequences of the definition and the summation properties developed in Section 8.4. The usefulness and the importance of these properties should again be stressed at this point. Again, they are useful in computational problems, and one or two of these problems might be done in class at this point in the development.

Although the theorem at the end of this section is important, the teacher should not spend too much time on it. If necessary, as a "convincer" or motivational device to begin this theorem, students should calculate the sums of the deviations from the mean (for a relatively small amount of data) to see that the theorem actually does hold.

For enrichment purposes you may want to introduce at least two other "mathematical averages". They have interesting properties and there are interesting relations between them.

The geometric mean of a set of n numbers is the nth root of their product. For example, the geometric mean of the numbers 4 and 9 is $\sqrt{4 \times 9} = 6$. (You will notice that for the case of two numbers the geometric mean is what is more commonly called the mean proportional.)

The harmonic mean of a set of numbers is the reciprocal of the arithmetic means of the reciprocals of the numbers. For example, the harmonic mean of the two numbers 4 and 9 is

$$\frac{1}{\frac{\frac{1}{4} + \frac{1}{9}}{2}} = \frac{2}{\frac{1}{4} + \frac{1}{9}} = \frac{72}{9 + 4} = \frac{72}{13} = 5.5 \text{ (approx.)}$$

The harmonic mean is used in problems involving rates. For example:

(1) If a man drives a distance of 60 miles at an average rate of 30 mph and returns over the same way at 20 mph, what was his average rate of speed for the round trip?

(2) One man can do a job in 3 days and another can do it in 2 days. How long will it take the two men to do the job if they work together? What is the average of their rates of work?

Exercises can be given to find the arithmetic, geometric, and harmonic means of two or more numbers and to compare them.

A culminating problem for such enrichment work might be to pose the problem of proving the following theorem.

"Prove that if two positive numbers $a < b$ are given:

(1) The geometric mean of a and b is the geometric mean of their harmonic mean and their arithmetic mean. (or $GM = \sqrt{HM \cdot AM}$)

(2) $a < HM < GM < AM < b$."

Two interesting discussions that can be developed about the concept of averages are: (1) What are the qualities that an average should have? (2) Which average is the best?

Some criteria for an average that might be cited are: it should be near the center of the distribution; it should be easy to compute and comprehend; it should depend in some way on all the measures; it should be stable with respect to grouping; etc.

The discussion of averages might elicit situations in which an average is not called for. For example: The height of a bridge over a river; the weight capacity of an elevator; comparing two cities by means of their mean annual temperatures; etc.

The general topic of statistics, its uses and misuses can give rise to some valuable discussions. Some good references for this topic are:

Wallist Roberts	The Nature of Statistics (Collier Books)--paperback
(Chapter 4:	Misuses of Statistics)
Mitchell and Walker	Algebra - A Way of Thinking (Harcourt, Brace and Co.)
(Chapter XI:	Statistics)
Walker and Lev	Elementary Statistical Methods

8.7 Exercises

These exercises are primarily designed to give students practice in working with means, and to broaden their understanding of the meaning and the properties of the mean. The first five exercises involve transforming sets of data and the effects of these transformations on the means, and should be done by all students. Problems 4 and 5 lead to the generalization asked for in Problem 7. All students should try to make this generalization, but not all should be expected to prove it. The teacher should point out again the importance of the summation material in doing this problem and Problem 19.

Problems 8-14 are largely computational and a selection of these should be assigned for homework. Problem 17 is a review problem. Problem 18 should be given, as it requires some thought about the basic definition of the mean and a little algebra. Problem 19 may be considered optional, although better students will profit by trying it. In a good class this problem might be given extra attention.

Answers to Exercises

1. Mean of 1st set of data = 5.
   Mean of new set of data = 12.
2. Mean of new set of measurements = 10.
   The mean of the new set of measurements is the sum of the means of the original sets.
3. Mean of 1st set = 5. Mean of new set = 35. The mean of the new set is 7 times the mean of the 1st set.
4. Mean of 1st set = 6. Mean of new set = 47. The mean of the new set is 5 more than 7 times the mean of the 1st set.
5. a. 38.3 b. 101 (to the nearest degree)
6. Mean = 35.4, Median = 36.
   Watch for confusion on wording here. Merely find the mean of the distribution and the median. "Number of students per class" and "number of students" are not to be construed as being special problems.

7. If M_d = mean of domain; M_r = mean of range, then

$M_r = CM_d + h$.

Proof: If we let $y_i = CX_i + h$, then

$$M_r = \frac{1}{n}\sum_{i=1}^{n} y = \frac{1}{n}\sum_{i=1}^{n}(CX_i + h)$$

$$= \frac{1}{n}\left(C\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} h\right) \qquad \text{by Problems 3 and 4, Section 8.5}$$

$$= \frac{1}{n}\left(C\sum_{i=1}^{n} X_i + hn\right) \qquad \text{by Problem 6, Section 8.5}$$

$$= C \cdot \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n}(nh) \qquad \text{by properties of real numbers (assoc., commut., etc.)}$$

$$= CM_d + h \qquad \text{by properties of real numbers, definition of } M_d.$$

8. The mean of the 80 numbers is 1093.6.

   a. The means of the 10 rows are 1078, 960, 1158, 1172, 1048, 1044, 1106, 1208, 1106, 1056. The mean of these 10 numbers is 1093.6.

   b. An easier method might be to average the 5 columns, then find the average of these 5 numbers.

9. Mean = 34.75

10. Answers will vary.

11. The mean of the ungrouped data (to the nearest tenth) is 56.1. From the frequency table, a mean may be calculated by the method of Example 2. Answers will vary slightly

but should be close to the mean calculated for the ungrouped data.

12. Answers will vary.

13. The mean (to the nearest 10th) is 41.5. The median is 45.

14. Mean = 5.65

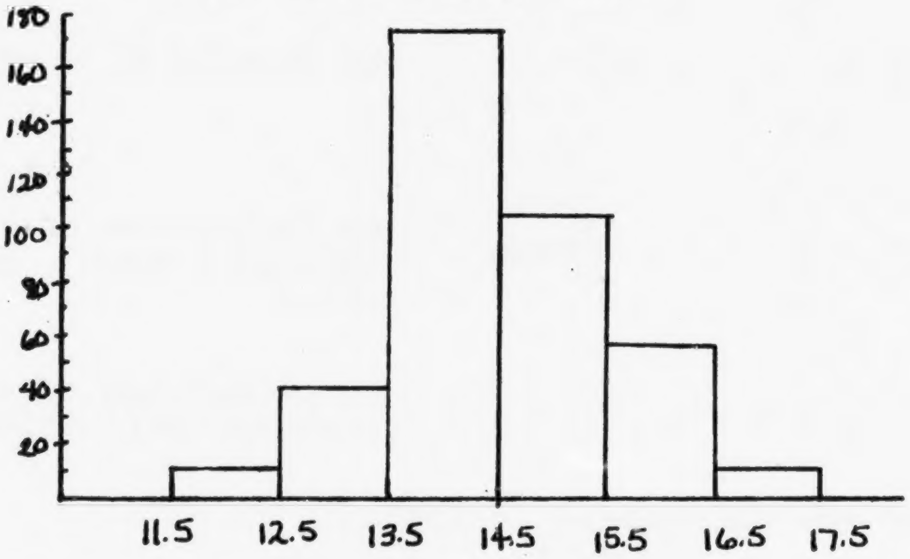
15. Mean = $75.1\overline{66}$

16. a. \$14,500 b. \$96.67

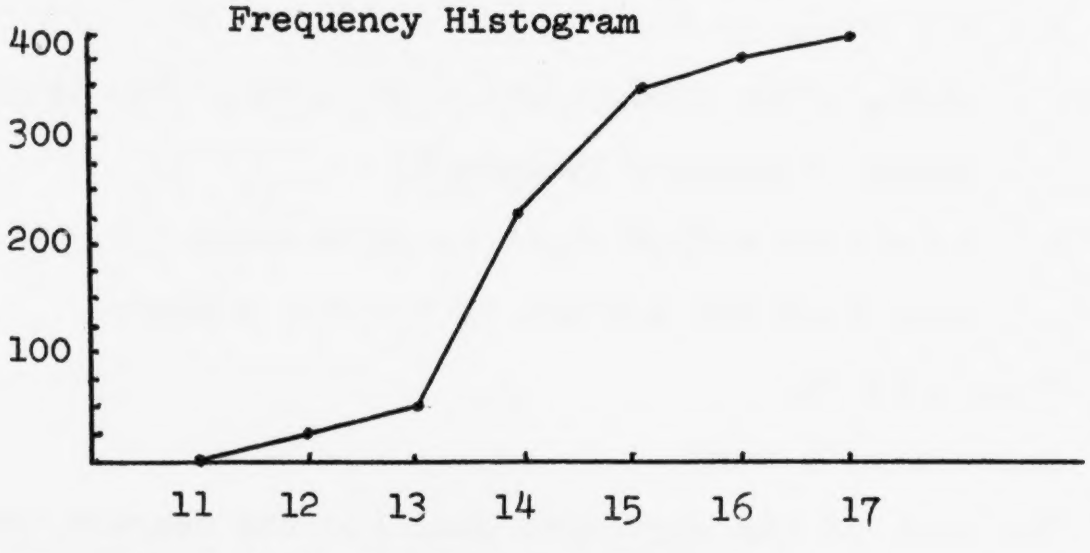
c. No, because the frequencies are different

17. a. 11.5 - 12.5, 12.5 - 13.5, etc.

b.



Frequency Histogram



Cumulative Frequency Polygon

18. The sum of the 6 observations given is 2104. If the missing observation is x, then $\frac{2104 + x}{7} = 350$.

Solving, $x = 346$.

19. $\sum (x_i - \bar{x})^2 + n(\bar{x} - a)^2 = \sum (x_i^2 - 2x_i\bar{x} + \bar{x}^2) + n(\bar{x}^2 - 2\bar{x}a + a^2)$

$$= \sum x_i^2 - 2\bar{x}\sum x_i + n\bar{x}^2 + n\bar{x}^2 - 2n\bar{x}a + na^2$$

(recall $\sum_{i=1}^{n} x_i = n\bar{x}$) $\quad = \sum x_i^2 - 2n\bar{x}^2 + 2n\bar{x}^2 - 2n\bar{x}a + na^2$

$$= \sum x_i^2 - \sum 2x_ia + \sum a^2$$

$$= \sum (x_i^2 - 2x_ia + a^2) = \sum (x_i - a)^2$$

8.8 Measures of Dispersion (Time estimate for 8.8 and 8.9 = 3 days)

The purpose of this section is to lead up to and then introduce the variance and standard deviation by first demonstrating a need for a statistic for measuring dispersion of data and then examining various possibilities for meeting this need. Before the students have read the text material, it might be desirable to state the problem (relying on the examples of Figure 8.7) and ask them for a proposed solution. Indicating the mean in color may be helpful. At this time it is important to discuss using the sum of the deviations from the mean as one possible solution. Before giving a formal treatment as in the text (using Theorem 1 to show that this sum is 0), the teacher might use the following informal diagrammatic approach: Using, for example, Figure 8.7 (b), indicate the mean (5) in color.

Now connect by arrows the symmetric pairs of data; e. g. 4 and 6 are symmetric and 3 and 7 are symmetric. Since the elements of each pair are the same distance from the mean, but in opposite directions, the total deviation of each pair is zero. In other words, each pair "balances out". Hence the sum of all the deviations is 0. For non-symmetric data it is not as easy to show in this way, but the argument is essentially the same: The sum of the left-hand distances from the mean equals the sum of the right distances from the mean, so the sum of the deviations must be 0. If this is the approach used, it should of course be followed by the formal treatment of the text. If Theorem 1 was covered lightly before, this would be a good point to go over the proof.

At some point either at the end of this section or of the next, the teacher should check to make sure that the students have a good intuitive understanding of what these new statistics are describing in contrast to those that have already been studied (that is, of dispersion versus central tendency).

Note that the students will need some knowledge of square root to understand and compute the standard deviation. Section 14.7, Course I describes one convenient algorithm that may be used to find square root. This can be taught quickly if students are not already familiar with it.

8.9 Exercises

These exercises will give the student computational practice with variances and standard deviations and will lead him to discover properties for these statistics analogous to properties he has already found for the mean--namely, how the variance and standard deviation are affected by adding or multiplying the data by constants. As the first three problems are entirely computational, the teacher may choose to omit one of them, depending on the class. Problems 4 and 5 are discovery exercises leading to the generalization asked for in Problem 6 (that adding a constant to the data does not change the variance). All students should do either 4 or 5 and should try Problem 6, and 6 should be gone over in class. Similarly, Problem 7 is a discovery exercise leading to the generalization of Problem 8 (multiplying the data by a constant will result in multiplying the variance by the square of that constant). All students should do 7 and try 8.

The results of Problems 6 and 8 may be demonstrated intuitively. If a constant is added to the data, the data is merely "moved over" on the axis. The mean is increased by the amount the data is moved--the additive constant--and the dispersion, and hence the variance, is unchanged. Under multiplication by a constant, the data is spread proportionately--it becomes more (or less) dispersed--so that the standard deviation will change proportionately.

Answers to Exercises

1.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	
measure	4,5,5, 5,6	3,4, 6,7	3,4,5, 6,7	4,4, 6,6	1,2,2, 8,8,9	8,9,10, 10,13	1,5,10, 16,18	1,4,10 11,14	2,5,8, 12,13	1
n	5	4	5	4	6	5	5	5	5	2
$\bar{x}$	5	5	5	5	5	10	10	8	8	3
range	2	4	4	2	8	5	17	13	11	4
R/n	.4	1.0	.8	.5	1.33	1.0	3.4	2.6	2.2	5
$x_i-\bar{x}$	-1,0,0, 0,1	-2,-1 1,2	-2,-1, 0,1,2	-1,-1, 1,1	-4,-3,-3 3,3,4	-2,-1,0 0,3	-9,-5 0,6,8	-7,-4 2,3,6	-6,-3 0,4,5	6
$\sum\|x_i-\bar{x}\|$	2	6	6	4	20	6	28	22	18	7
$\frac{\sum\|x_i-\bar{x}\|}{n}$	.4	1.5	1.2	1.0	3.33	1.2	5.6	4.4	3.6	8
$(x_i-\bar{x})^2$	1,0,0, 0,1	4,1,1, 4	4,1,0, 1,4	1,1, 1,1	16,9,9, 9,9,16	4,1,0, 0,9	81,25,0 36,64	49,16 4,9,36	36,9,0 16,25	9
$\sum(x_i-\bar{x})^2$	2	10	10	4	68	14	206	114	86	10
Variance s^2	.4	2.5	2.0	1.0	11.3	2.8	41.2	22.8	17.2	11
Standard Deviation	.63	1.6	1.4	1.0	3.4	1.7	6.4	4.8	4.1	12

2. a. Mean = $(22 + 26 + 20 + 31 + 26) + 5 = 125/5 = 25$.

   Median 26.

   b. Mean absolute deviation = $(3 + 1 + 5 + 6 + 1) + 5 = 16/5 = 3.2$.

   c. Variance = $\dfrac{(-3)^2 + (1)^2 + (-5)^2 + (6)^2 + (1)^2}{5} = 14.4$

   d. Standard Deviation = $\sqrt{14.4} = 3.8$ approx.

3. The measures are: 347, 351, 358, 345, 350, 353, and 346.

   Deviations from the mean are -3, +1, +8, -5, 0, +3, -4.

   The squares of the deviations are 9, 1, 64, 25, 0, 9, 16.

   The sum of the squares of the deviation is 124.

   The variance = $124/7 = 17.71$

   The standard deviation = $\sqrt{17.71} = 4.21$ approx.

4. Mean = $(8 + 10 + 24)/3 = 42/3 = 14$

   Variance = $\dfrac{(-6)^2 + (-4)^2 + (10)^2}{3} = \dfrac{152}{3} = 50.66$

   Standard deviation = $\sqrt{50.66} = 7.12$ approx.

   a. 5,7,21 Mean = $\dfrac{5 + 7 + 21}{3} = 11$

   Variance = $\dfrac{(-6)^2 + (-4)^2 + (10)^2}{3} = \dfrac{152}{3} = 50.66$

   Standard Deviation = 7.12 approx.

b. The mean is decreased by 3. The variance and the standard deviation remain the same.

5. Mean = $(1 + 6 + 8)/3 = 15/3 = 5.$

Variance = $\frac{(-4)^2 + (1)^2 + (3)^2}{3} = \frac{16 + 1 + 9}{3} = \frac{26}{3} = 8.66$

The variance of the new measures will be 8.66.

6. To show that the variance of a set of measurements is not changed if a constant is added to each measurement.

Given: Set x_i of measurements

Let $y_i = x_i + k$

Now $s_x^2 = \frac{1}{n}\sum(x_i - \bar{x})^2$ (1)

and $s_y^2 = \frac{1}{n}\sum(y_i - \bar{y})^2$ (2)

We already know that if

$$y_i = x_i + k$$

Then

$$\bar{y} = \bar{x} + k$$

Let us substitute these values in (2)

$$s_y^2 = \frac{1}{n}\sum(x_i + k - (\bar{x} + k))^2$$

$$= \frac{1}{n}\sum(x_i - \bar{x})^2 = s_x^2$$

7. The mean of the numbers -1, -3, 1 is -1.

The mean of the original set is 9000 - 1 = 8999.

The deviations from the mean are 0, -2, +2.

The squares of the deviations are 0, 4, 4.

The sum of these is 8.

The variance = 8/3 = 2.66.

The variance of the original set of measurements is 2.66.

8. To show that if each of a set of observations is multiplied by k (a) The variance is multiplied by k^2, (b) the standard deviation is multiplied by k.

Given: The measurements x_i and $y_i = kx_i$.

To show: $s_x^2 = s_x^2 k^2$

$$s_x^2 = \frac{1}{n}\sum (x_i - \bar{x})^2$$

$$s_y^2 = \frac{1}{n}\sum (y_i - \bar{y})^2$$

We already know that if $y_i = kx_i$ then $\bar{y} = k\bar{x}$

Substituting, $s_y^2 = \frac{1}{n}\sum (kx_i - k\bar{x})^2$

$$= \frac{1}{n}\sum k^2 (x_i - \bar{x})^2$$

$$= k^2 \cdot \frac{1}{n}\sum (x_i - \bar{x})^2 = k^2 \cdot Sx_x^2 .$$

8.10 Simplified Computation of the Variance and the Standard Deviation (Time estimate for 8.10 and 8.11 = 2 days)

After having completed the previous set of exercises, the students should be eager to learn some short-cut devices for calculating the variance and standard deviation. The teacher should do carefully, in class, the derivation given in this section. It might be desirable to begin this material before the students have read it, and to try to get them to supply the main line of the argument. Start with the definition of the standard deviation and ask the students to expand it. There is a natural procedure to follow in proofs of this type, and hopefully the better students will have begun to pick it up.

It might also be desirable to work out (in class, as an illustration) one of the previous homework exercises using the shortcut.

8.11 Exercises

Students should be expected to do at least 2 of these problems. The solution to Problem 1 is given in the text. Solutions to Problems 2 and 3 will vary; however, at least one solution should be gone over in class.

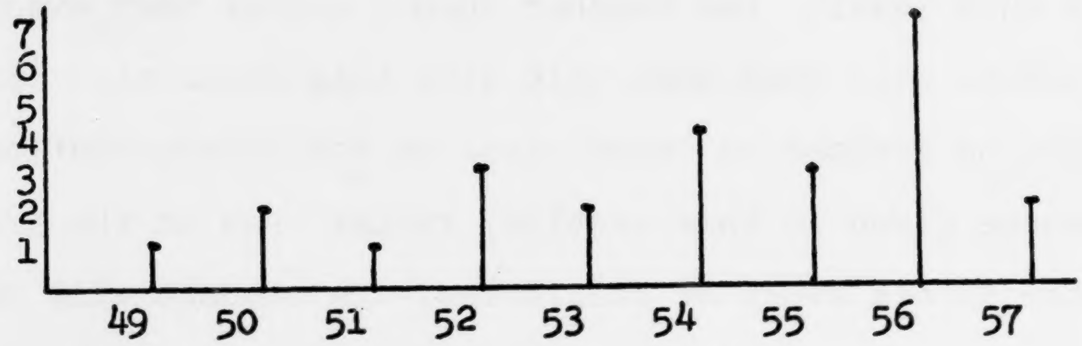
8.12 The Chebyshev Inequality

In any but an exceptional class, it would probably be best to regard this section as enrichment material for the best students. The proof of the Chebyshev Inequality involves some manipulations and statements that are quite sophisticated for this level. The teacher should expect that even very good students will need some help with this material. Students might be advised to focus first on the interpretation of the theorem given in this section, rather than on the proof. Later in his study of statistics, the student will see that this theorem can be improved upon for certain assumed distributions; for example, if the data is assumed to follow a normal frequency distribution, approximately two-thirds of the data will lie within one standard deviation of the mean, and about 95% will lie within two standard deviations of the mean.

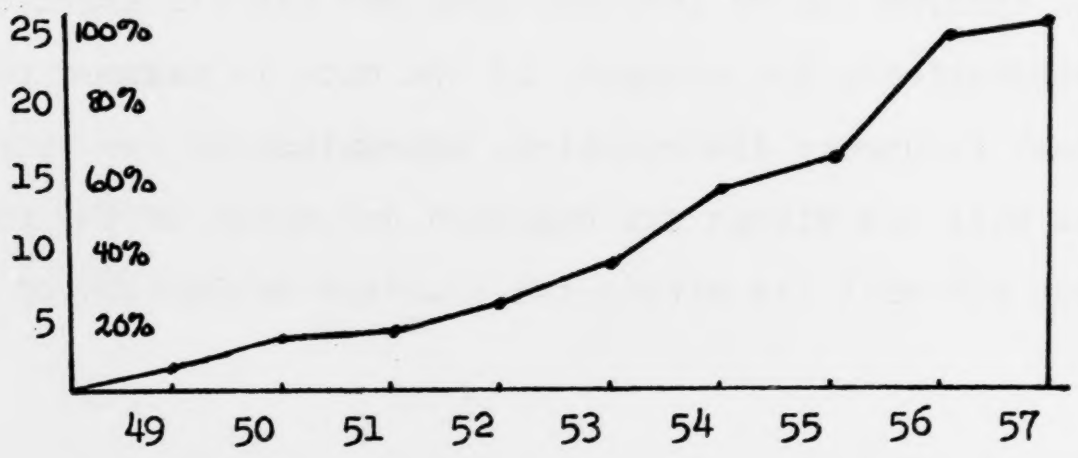
Note that the Chebyshev Inequality, in contrast, states only that not all of the data will be farther than one standard deviation from the mean; further that at least 3/4 of the data will lie within two standard deviations.

8.14 Review Exercises

1. a.



b.



c. Median = 54. Lower quartile = 52. Upper quartile = 56.

d. The mode is 56.

e. The range is 57 - 49 = 8

The interquartile range is 56 - 52 = 4

f, g, h.

x_i	f_i	$y_i = x_i - 53$	$x_i f_i$	$f_i y_i$	$f_i x_i^2$	$f_i y_i^2$
49	1	-4	49	-4	2401	16
50	2	-3	100	-6	5000	18
51	1	-2	51	-2	2601	4
52	3	-1	156	-3	8112	3
53	2	0	106	0	5618	0
54	4	1	216	4	11664	4
55	3	2	165	6	9075	12

56	7	3	392	21	21952	63
57	2	4	114	8	6498	32
$\sum$	25		1349	24	72921	152

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1349}{25} = 53.96$$

$$\bar{x} = \bar{y} + 53 \qquad \bar{y} = \frac{24}{25} = .96$$

$$\bar{x} = .96 + 53 = 53.96$$

$$s_x^2 = \frac{\sum f_i x_i^2}{\sum f_i} - \bar{x}^2$$

$$= \frac{72921}{25} - 2911.6816$$

$$= 2916.84 - 2911.6816$$

$$= 5.1584$$

$$s_x^2 = s_y^2 = \frac{152}{25} - (.96)^2 = 5.1584$$

$$s_x = s_y = \sqrt{5.1584} = 2.27$$

2. a. The range is 4300

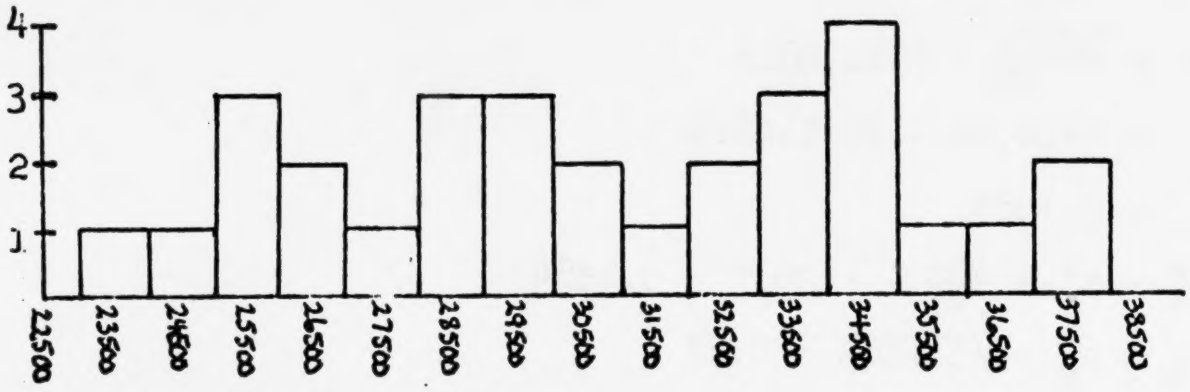
b.

Interval Boundaries	Midpoints x_i	Frequency f_i	Cum. Freq.
23,000 - 24,000	23,500	1	1
24,000 - 25,000	24,500	1	2
25,000 - 26,000	25,500	3	5
26,000 - 27,000	26,500	2	7
27,000 - 28,000	27,500	1	8
28,000 - 29,000	28,500	3	11
29,000 - 30,000	29,500	3	14
30,000 - 31,000	30,500	2	16
31,000 - 32,000	31,500	1	17
32,000 - 33,000	32,500	2	19
33,000 - 34,000	33,500	3	22
34,000 - 35,000	34,500	4	26
35,000 - 36,000	35,500	1	27
36,000 - 37,000	36,500	1	28
37,000 - 38,000	37,500	2	30

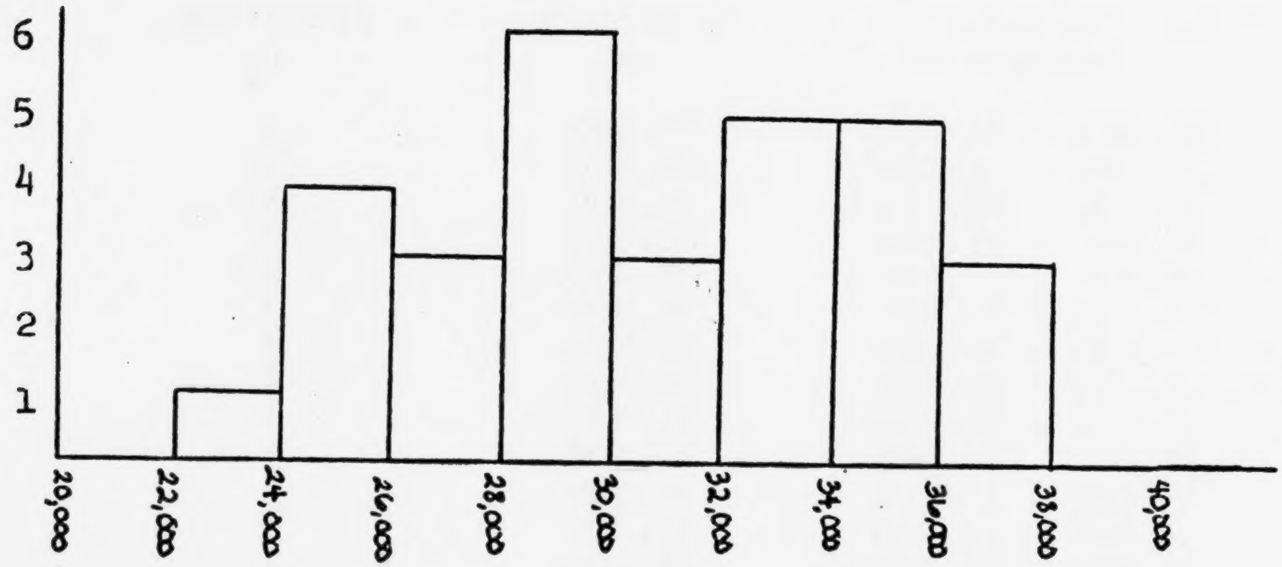
Interval Boundaries	Midpoints x_i	Frequency f_i	Cum Freq.
22,000 - 24,000	23,000	1	1
24,000 - 26,000	25,000	4	5
26,000 - 28,000	27,000	3	8
28,000 - 30,000	29,000	6	14
30,000 - 32,000	31,000	3	17
32,000 - 34,000	33,000	5	22
34,000 - 36,000	35,000	5	27
36,000 - 38,000	37,000	3	30

c. In the first grouping the mode is 34,500. In the second grouping the mode is 29,000.

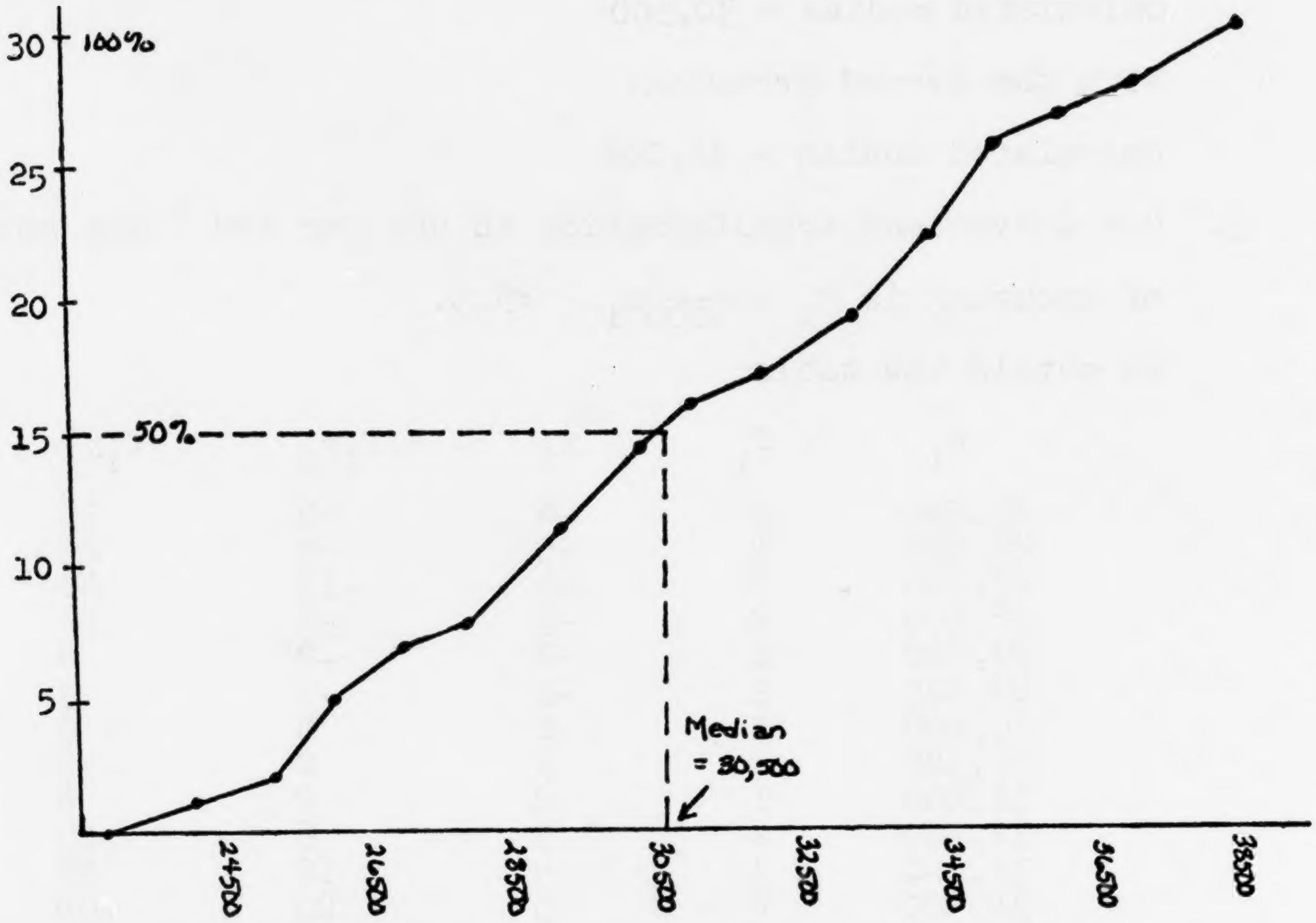
d. With the first grouping:



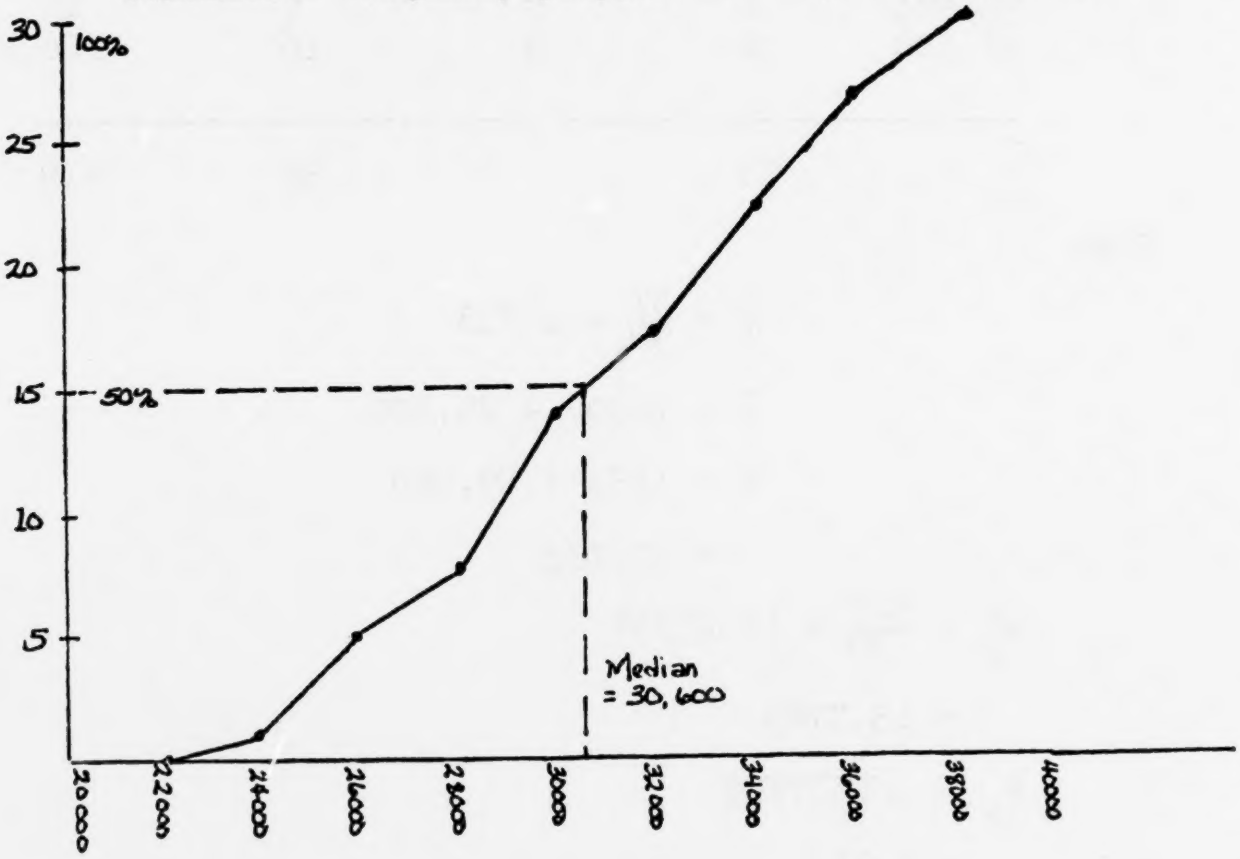
With the second grouping:



e. Cumulative frequency polygon with first grouping:



Cumulative frequency polygon with second grouping:



f. With the first grouping:

Calculated median = 30,500

With the second grouping:

Calculated median = 31,000

g. One convenient transformation to use for the first method of grouping is $y_i = \frac{1}{1000}x_i - 29.5$.

We obtain the table:

x_i	f_i	y_i	$f_i y_i$	$f_i y_i^2$
23,500	1	-6	-6	36
24,500	1	-5	-5	25
25,500	3	-4	-12	48
26,500	2	-3	-6	18
27,500	1	-2	-2	4
28,500	3	-1	-3	3
29,500	3	0	0	0
30,500	2	1	2	2
31,500	1	2	2	4
32,500	2	3	6	18
33,500	3	4	12	48
34,500	4	5	20	100
35,500	1	6	6	36
36,500	1	7	7	49
37,500	2	8	16	128
$\sum$	30		37	519

Then

$$\bar{y} = \frac{37}{30} = 1.233$$

$$\bar{x} = 1000\bar{y} + 29{,}500$$

$$\bar{x} = 1233 + 29{,}500$$

$$= 30{,}733$$

$$s_y^2 = \frac{519}{30} - (1.233)^2$$

$$= 15.7789$$

$$s_y = \sqrt{15.7789}$$

$$= 3.97$$

$$s_x^2 = (1000)^2 \cdot s_y^2$$

$$= 15{,}778{,}900$$

$$s_x = 1000 \cdot s_y$$

$$= 3970$$

In the second method of grouping we might make the transformation

$$y_i = \frac{1}{2000}x_i - \frac{31}{2}$$

x_i	f_i	y_i	f_iy_i	$f_iy_i^2$
23,000	1	-4	-4	16
25,000	4	-3	-12	36
27,000	3	-2	-6	12
29,000	6	-1	-6	6
31,000	3	0	0	0
33,000	5	1	5	5
35,000	5	2	10	20
37,000	3	3	9	27
$\sum$	30		-4	122

Then

$$\bar{y} = \frac{-4}{30} = -.133$$

$$\bar{x} = 2000\bar{y} + 31{,}000$$

$$= -266 + 31{,}000$$

$$= 30{,}734$$

$$s_y^2 = \frac{122}{30} - (-.133)^2$$

$$= 4.0666 - .0177$$

$$= 4.0489$$

$$s_y = 2.01$$

$$\begin{aligned} s_x^2 &= (2000)^2(s_y^2) \\ &= (4{,}000{,}000)(4.0489) \\ &= 16{,}195{,}600 \\ &= (2000)(s_y) \\ &= 4020 \end{aligned}$$

Chapter 9

Transformations In The Plane: Isometries

Time Estimate For Chapter: 25 - 28 days

This Chapter continues our study of geometry. The background for this continuation is considerable. Students have had experiences with proofs and some appreciation of the nature of an axiomatic system. Of great importance to our purposes in this Chapter are coordinates, for they will make possible a simple presentation and a facile method of proof. However, a continued reliance on coordinates leave some students with the impression that the use of coordinates is mechanical, and this is unfortunate. This Chapter should help to remove this unhappy impression.

When Euclid proved his triangle congruence principles he moved triangles about to make one coincide with another. He must have been uneasy about those motions for he does not use this technique as a regular practice. Besides, he must have known he was moving something that was not physical. Yet in these motions lie the beginnings of the subject matter of this Chapter, which studies the nature and properties of "rigid motions", the mathematical counterparts of physical motions. We recognize four such "motions", reflections in a line (symmetry in a line) translation, rotations about a point (including a half-turn about a point or symmetry in a point) and glide reflections. It is essential to regard these as special cases of transformations, that is, mathematical one-to-one mappings of the entire place onto itself.

The set of rigid motions constitute a group under the operation of composition. They differ from other groups of transformations in their preservation of distance. We call the set of rigid motions the set of isometries.

A word of caution concerning rigor. While we want students to understand what rigor means, we can't expect them to conduct themselves as though they were professional mathematicians who are publishing for other professional mathematicians. Chapter 4 and the first parts of Chapter 6 are occupied with teaching the concept of rigorous procedures. In this Chapter we relax standards of rigor. Occasionally we rely on pictures; sometimes we omit consideration of special cases in the interest of avoiding the practice of "nit-picking" which can be so deadly to the interest of most students.

Another word of caution. There are a great variety of transformations. Our main concern is with plane isometries. But it would be unfortunate to leave the impression that isometries are the only kind of transformation. We try to dispel this impression in Section 9.19, where we present some elementary notions about dilations (homotheties) and similarities. Other types of transformations not mentioned in this Chapter are perspective affinites, affine transformations, prejective transformations and topological transformation. These are studied in advanced grades.

And a final word of caution. After students have studied isometries they will have two main methods of proof: using coordinates or using isometries. They should be encouraged to develop skill in the use of both methods. It is unfortunate that success in the use of one of those methods tends to discourage use of the other.

A word about the general structure of this Chapter. After explaining what plane transformations are (Section 9.1) the special cases of isometries are introduced (Sections 9.3, 9.5, 9.7, 9.9). These presentations are guided by two objectives. (1) To show that all isometries are compositions of line reflections (three at the most). (2) The set of isometries, under the operation of composition, is a group. This overall view may help to explain some of the exercises in the early exercise section. For instance it is not too early to look at the group features at the outset long before the word "group" is used. The student is reminded early that composition of mappings is associative, from which it follows that the composition of transformations is associative. The identity transformation must also be recognized quite early. This too helps to prepare the student for an understanding of groups of transformations and subgroups.

References

1. Max Jeger: Transformation Geometry
John Wiley and Sons Translated by Deike
2. I. M. Yaglom: Geometric Transformations
The L. W. Singer Co.: Translated by Allen Schnields 1962

3. B. V. Kutuzov: Studies in Mathematics, Vo. IV.
Geometry School Mathematics Study Group. 1960
Translated by L.I. Gordon and E. S. Shater

9.1 What is a transformation? (1 - 2 days)

It is advisable to review the meaning of mapping and operational system.

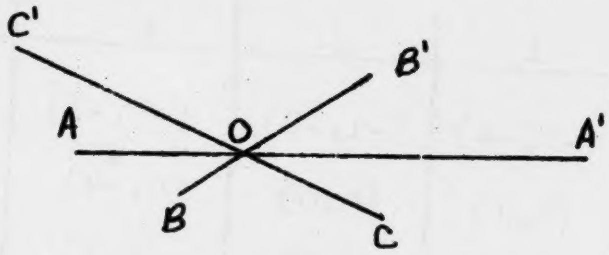
Since composition of maps F and G is possible when the range of F is equal to the domain of G, and the domain and range of all our transformations are the set of points of a plane, it is simple matter to compose two transformations in a plane. It is a short step to conclude that composition of plane transformations is associative, for we have already concluded that composition of mappings is associative.

In studying some of the transformations students should be encouraged to look for properties that are preserved. These searches can be based on drawings. Of course the basic property that interests us is the one that preserves distance; that is, the distance between any two points is equal to the distance between their image points. An interesting transformation is found in Exercise 8 of Section 9.2. Under this transformation the distance between an infinite number of pairs of points is preserved, but for another infinite number it is not. Among other properties students should look for are preservation of collinearity, betweeness relation for points, parallelism between two lines, and measure of angles. It would be instructive to look for properties that do not belong to transformations.

Exercise 4 contains many exercises of mappings in terms of coordinate rules. Students should do most of them, if not all, for coordinate rules are going to be an important mathematical asset in this Chapter. Also they are interesting.

9.2 Exercise Solutions

1.



This mapping is a transformation because it's a mapping of the plane onto itself and is one-to-one. To locate the origin of D', take D such that D', O, D are collinear and $OD = \frac{1}{2} OD'$.

2. The figure above can be used if each letter and its "prime" are interchanged. To find the original of D' take D such that D', O, D are collinear and $OD = 2\ OD'$.

3. $A(-2, -1) \longrightarrow A'\ (1, -3)$

$B(0, 4) \longrightarrow B'\ (3, 2)$

$C(3, 2) \longrightarrow C'\ (6, 0)$

$D(1, -3) \longrightarrow D'\ (4, -5)$

This procedure is a mapping because a unique image is assigned to each point. It is a transformation because the mapping is onto the plane and is one-to-one. To find the original of an image use $(x, y) \longrightarrow (x-3, y+2)$.

4.

	a	b	c	d	e	f
A'	(-2,1)	(2,-1)	(2,1)	(4,-1)	(-4,1)	(-8,-1)
B'	(0,-4)	(0,4)	(0,-4)	(0,4)	(0,-4)	(0,4)
C'	(3,-2)	(-3,2)	(-3,-2)	(9,2)	(-9,-2)	(27,2)
D'	(1,3)	(-1,-3)	(-1,3)	(1,-3)	(-1,3)	(1,-3)

	g	h	i	j	k	l
A'	(-8,0)	(4,1)	(-3,-4)	(-1,-2)	(-3,-1)	(-3,-4)
B'	(0,5)	(0,16)	(1,1)	(4,0)	(4,-4)	(-4,8)
C'	(27,3)	(9,4)	(7,-1)	(2,3)	(5,1)	(4,7)
D'	(1,-2)	(1,9)	(3,-6)	(-3,1)	(-2,4)	(5,-5)

a) Transformation $P'(a, b) \longrightarrow P(a, -b)$

b) Transformation $P'(a, b) \longrightarrow P(-a, b)$

c) Transformation $P'(a, b) \longrightarrow P(-a, -b)$

d) Mapping but not a transformation - neither onto nor one-to-one

e) Mapping but not a transformation - neither onto nor one-to-one

f) Transformation $P'(a, b) \longrightarrow P(\sqrt[3]{a}, b)$

g) Transformation $P'(a, b) \longrightarrow P(\sqrt[3]{a}, b-1)$

h) Mapping but not a transformation - one-to-one but not onto

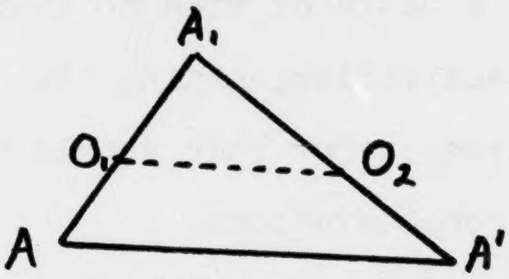
i) Transformation $P'(a, b) \longrightarrow P(\frac{a-1}{2}, b+3)$

j) Transformation $P'(a, b) \longrightarrow P(b, a)$

k) Transformation $P'(a, b) \longrightarrow P(\frac{a+b}{2}, \frac{a-b}{2})$

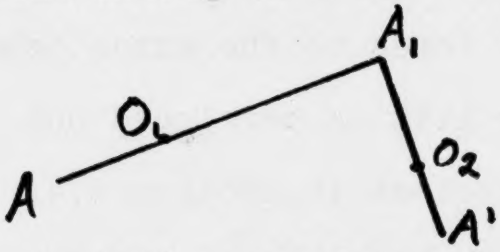
l) Transformation $P'(a, b) \longrightarrow P(\frac{2a-b}{5}, \frac{2b-a}{5})$

5.



This is both a mapping and a transformation. It is worth noting that for each point P, $PP' = 2O_1O_2$.

6.



It is both a mapping and a transformation. To find the original of P' take P_1 such that O_2 is the midpoint of $\overline{P'P_1}$, then find P (the original of P') such that P_1, O, and P' are collinear and $\overline{O_1P} = \frac{1}{2}\overline{O_1P_1}$.

7.

$$A \xrightarrow{gof} A_2 \xrightarrow{h} A_3$$

$$A \xrightarrow{f} A_1 \xrightarrow{hog} A_3$$

Since this is true for all A we may conclude that the composition of transformations in a plane is associative. (This result will be used later in the chapter in connection with a study of groups of transformations).

8. Yes. Assignments are unique; the mapping is one-to-one and onto. Therefore it is a transformation. (Note. This transformation has none of the interesting properties belonging to isometries and similarities).

9.3 Reflections in a Line (1 - 2 days)

Students have already had a look at these transformations in Course 1. But it would not be a waste of time to introduce them again through paper folding activities, noting its characteristics and what it preserves. But this should be done quickly and lead to mathematical considerations.

We spend much time to establish the distance preserving property of reflections, for this stamps them as isometries. But more than this, this property leads to the establishment of a list of other properties. This list is developed out of students experiences with the exercises in Section 9.4. Being exercises should not diminish their importance, and for this reason a summary of these properties appears at the end of that section. Your students may want to refer to that summary more than once. Most important is the fact that these properties may be "inherited" by all other isometries, for it will be shown that other isometries can be regarded as a composition of line reflections. Thus any property that survives in such a composition belongs to the composition.

Reflections in lines which are not parallel to an axis will be studied in Course III. Reflections in line $y = x$ or $y = -x$ may be tried in Course II if desired.

9.4 Exercise Solutions:

1.	(a)	$(3, -5)$	(b)	$(-3, -5)$	(c)	$(-3, 5)$
	(d)	$(\sqrt{2}, 5)$	(e)	$(3, 0)$	(f)	$(0, 3)$
	(g)	$(0, 0)$	(h)	$(a, -b)$		

2. (a) $(-3, 5)$ (b) $(3, 5)$ (c) $(3, -5)$
(d) $(-\sqrt{2}, -5)$ (e) $(-3, 0)$ (f) $(0, -3)$
(g) $(0, 0)$ (h) $(-a, b)$

3. (a) $2 = \frac{1}{2}(3 + x_2) \longrightarrow x_2 = 1 \therefore A'(1, -2)$
(b) $2 = \frac{1}{2}(-2 + x_2) \longrightarrow x_2 = 6 \therefore B'(6, 5)$
(c) $2 = \frac{1}{2}(\sqrt{2} + x_2) \longrightarrow x_2 = 4 - \sqrt{2} \therefore C'(4 - \sqrt{2}, 3)$
(d) $2 = \frac{1}{2}(a + x_2) \longrightarrow x_2 = 4 - a \therefore D'(4 - a, b)$

4. (a) $-3 = \frac{1}{2}(3 + x_2) \longrightarrow x_2 = -9 \therefore A'(-9, -2)$
$-3 = \frac{1}{2}(-2 + x_2) \longrightarrow x_2 = -4 \therefore B'(-4, 5)$
(b) $1 = \frac{1}{2}(-2 + y_2) \longrightarrow y_2 = 4 \therefore A'(3, 4)$
$1 = \frac{1}{2}(5 + y_2) \longrightarrow y_2 = -3 \therefore B'(-2, -3)$
(c) $-2 = \frac{1}{2}(-2 + y_2) \longrightarrow y_2 = -2 \therefore A'(3, -2)$
$-2 = \frac{1}{2}(5 + y_2) \longrightarrow y_2 = -9 \therefore B'(-2, -9)$

5. (a) $A(4, 2) \xrightarrow{Ry} A'(-4, 2),\ B(-1, 5) \xrightarrow{Ry} B'(1, 5)$
$AB = \sqrt{(4+1)^2 + (2-5)^2} = \sqrt{34}$
$A'B' = \sqrt{(-4-1)^2 + (2-5)^2} = \sqrt{34}$
(b) $A(0, 5) \quad \overset{Ry}{} \quad A'(0, 5),\quad B(4, -1) \quad \overset{Ry}{} \quad B'(-4, -1)$
$AB = \sqrt{(0-4)^2 + (5+1)^2} = \sqrt{52}$
$A'B' = \sqrt{(0+4)^2 + (5+1)^2} = \sqrt{52}$
(c) $A(-2, 0) \xrightarrow{Ry} A'(2, 0),\ B(0, -5) \xrightarrow{Ry} B'(0, -5)$
$AB = \sqrt{(-2-0)^2 + (0+5)^2} = \sqrt{29}$
$A'B' = \sqrt{(2-0)^2 + (0+5)^2} = \sqrt{29}$

6. AB + BC = AC because B is between A and C.
AB = A'B', BC = B'C', AC = A'C' because R_ℓ is an isometry.
A'B' + B'C' = A'C' because of the substitution property of =.

(a) If B' is not in $\overleftrightarrow{A'C'}$ then A', B' C' are vertices of a triangle and by the triangle inequality theorem (See Course 1 Chapter 10) $A'B' + B'C' > A'C'$.

(b) If A' is between B',C' then $B'A' + A'C' = B'C'$, or $A'C' = B'C' - B'A'$. But $A'C' = A'B' + B'C'$. $\therefore A'B' = 0$ or A' and B' are not distinct points. This implies that A and B are the same point - a contradiction. Similarly, C' cannot be between A' and B' (we have not considered the case $A' = B'$ or $C' = B'$, for this leads easily to $A = B$ or $C = B$).

7. By Exercise 6 the betweeness relation for points is preserved under a line reflection. It follows immediately that collinearity is preserved since the betweeness relation implies collinearity.

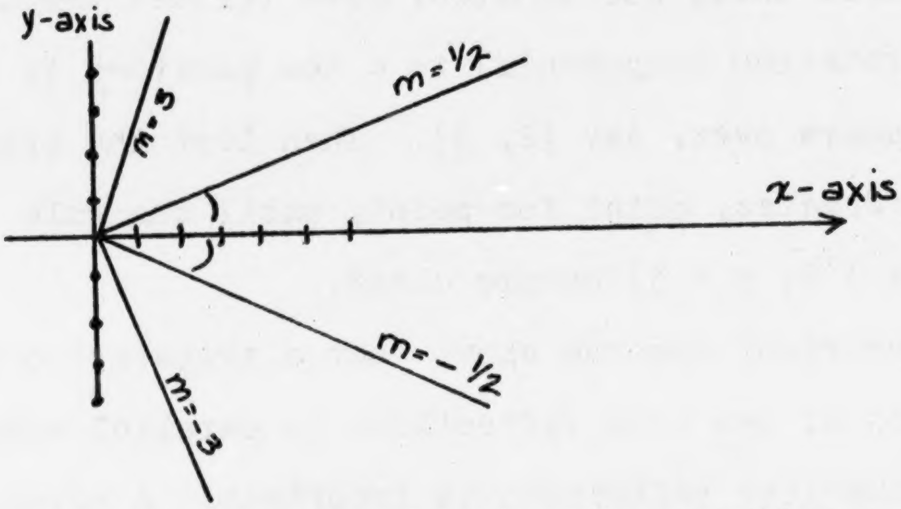
8. The reflection of the endpoint of a ray is a point. For any three points of the ray, of which one is the endpoint, their images under a line reflection have the same betweeness relation as their originals. Therefore the images of all interior points of the ray are collinear and on the same side of the image of the endpoint. Therefore the image of a ray is a ray.

   A similar argument can be made, applying to both endpoints of a segment. Hence the image of a segment under a line reflection is a segment.

9. (a) By Exercise 8 the images of the sides of $\angle AOB$ are rays having the same endpoint, $\overrightarrow{O''A''}$ and $\overrightarrow{O'B'}$. Let C

be any point between A and B. We may assert that $\overrightarrow{OC}$ is a ray of $\angle AOB$. The image C' of C is between A' and B'. Therefore $\overrightarrow{O'C'}$ is a ray between $\overrightarrow{O'A'}$ and $\overrightarrow{O'B'}$. Hence the image of $\angle AOB$ is $\angle A'O'B'$.

(b)



10. If ℓ is the x-axis, ℓ_1 may have the equation y = k, ℓ_2 the image of ℓ_1, under R_x, is y = -k. Therefore $\ell_1 \parallel \ell_2$.

11. Let ℓ be the x-axis of a rectangular coordinate system. Then ℓ_1 may have equation of form x = k. Since $(x, y) \xrightarrow{Rx} (x, -y)$. The image of x = k is x = k. Therefore $\ell_1 = \ell_2$.

12. Using a rectangular coordinate system with ℓ as x-axis we may assign (a, b) to P. Then P' has coordinates (a, -b). The image of P' under R_x is (a, -(-b)) or (a, b). Therefore $P = P_1$.

9.5 Translations (2 - 3 days)

The first step in presenting (reviewing) this transformation can be taken with the aid of an overhead projector. Let the base be a coordinate plane, and prepare a transparency which is also coordinated by the same set of lines as the base. (The axis numerals should appear only on the base). Start the demonstration with the transparency coinciding with the base, mark a set of points on the transparency (include the origin), mark their coordinates, move (glide) the transparency (with no rotation components) to a new position in which (0, 0) appears over, say (2, 3). Then list the original and image coordinates, point for point, until the rule $(x, y) \longrightarrow (x + 2, y + 3)$ become clear.

An important theorem shows that a translation is the composition of two line reflections in parallel axes. The order of the line reflection is important. A reversal of order results in a different translation, in fact, the inverse translation. Another amazing feature is that any pair of axes can be used, if they satisfy the direction and distance requirements, to produce the same translation.

Thus a given translation can be regarded as the composition of an infinite number of pairs of line reflections, in parallel axes. You may have to clarify what is meant by the distance between two parallel lines.

The section ends in a mass inheritance from the list of properties of line reflections. This inheritance procedure will be repeated in discussing rotations and glide reflections.

9.6 Exercises

All problems are necessary for clear understanding and will be used as reference in later development. Prior to assignment, there should be discussion of mathematical usage of the letters "x", "y", and "z" -- used as variables -- and the letters "a", "b", and "c" denoting specific numbers. In exercises 2(d), (e) and (f), you can really bring the usage to light through discussion of errors which will definitely crop up.

9.6 Exercise Solutions

1. (a) $(4, 3)$ (b) $(-2, 3)$ (c) $(-2, -1)$
(d) $(9\frac{1}{2}, 4\frac{1}{2})$ (e) $(1 + \sqrt{2}, 2)$ (f) $(0, 1 + \sqrt{3})$
(g) $(1, 1)$ (h) $(0, 2)$

2. (a) $(x, y) \longrightarrow (x-1, y+1)$ (b) $(x, y) \longrightarrow (x-3, y+5)$
(c) $(x, y) \longrightarrow (x+3, y-5)$ (d) $(x, y) \longrightarrow (x-5, y-1\frac{2}{3})$
(e) $(x, y) \longrightarrow (x-a, y-b)$ (f) $(x, y) \longrightarrow (x+a, y+2b)$

3. $A(0, 2) \longrightarrow A'(-1, 4)$, $B(5, 1) \longrightarrow B'(4, 3)$.

(a) A translation is an isometry and preserves distance $(\sqrt{1 + 2^2} = \sqrt{1 + 2^2})$.

(b) Under an isometry the image of a segment is a segment.

(c) Slope of $\overleftrightarrow{AB} = \frac{2-1}{0-5} = -\frac{1}{5}$. Slope of $\overleftrightarrow{A'B'} = \frac{1-4}{4-3} = -\frac{1}{5}$.

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$.

(d) Slope of $\overleftrightarrow{AA'} = \frac{2-4}{0+1} = -2$, slope of $\overleftrightarrow{BB'} = \frac{1-3}{5-4} = -2$.

$\therefore \overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$.

(e) $AA' = \sqrt{1^2 + 2^2}$, $BB' = \sqrt{1^2 + 2^2}$ $\therefore AA' = BB'$.

(f) Since the sides in each pair lie in parallel lines, ABB'A' is, by definition a parallelogram.

(g) The diagonals of a parallelogram bisect each other. Alternately, the midpoint of $\overline{AB'}$ has coordinates $(2, \frac{5}{2})$. The midpoint of $\overline{A'B}$ also has coordinates $(2, \frac{5}{2})$. For the general proof we may start with $AA' = BB'$ and $AA' \parallel BB'$. These conditions reflect the nature of a translation. If we assume that ABB'A' is a parallelogram if a pair of opposite sides have the same length and lie in parallel lines the proof is complete. Otherwise we may use a coordinate proof starting with $A(x_1, y_1)$, $B(x_2, y_2)$ and a translation with rule $(x, y) = (x+a, y+b)$. Then A' has coordinates $(x_1 + a, y_1 + b)$ and B' has coordinates $(x_2 + a, y_2 + b)$.

To prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$ use slopes.

Slope of $\overleftrightarrow{AB} = \frac{y_1 - y_2}{x_1 - x_2}$ and slope of $\overleftrightarrow{A'B'} = \frac{y_1 - y_2}{x1 - y_2}$

This suggests a consideration of two cases: (i) $x_1 = x_2$, in which case $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel to the y-axis and hence to each other and (ii) $x_1 \neq x_2$, in which case $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ have equal slopes and are parallel. To prove $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'}$ we work with slopes $\frac{b}{a}$ and $\frac{b}{a}$. $a = 0$, implies $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$ are parallel to the x-axis, etc.

4. $(x, y) \xrightarrow{T_1} (x+3, y-2) \xrightarrow{T_2} (x+4, y+1)$.

$(x, y) \xrightarrow{T_2 \circ T_1} (x+4, y+1)$

This shows that $T_2 \circ T_1$ is a translation.

$A(3, 2) \xrightarrow{T_2 \circ T_1} (7, 3)$

$B(-4, 0) \xrightarrow{T_2 \circ T_1} (0, 1)$

$C(-2, -5) \xrightarrow{T_2 \circ T_1} (2, -4)$

5. $(2, 3) \xrightarrow{T_1} (5, 1) \xrightarrow{T_2} (2, 3)$

Thus $(2, 3) \xrightarrow{T_2 \circ T_1} (2, 3)$

This suggests that $T_2 \circ T_1 = i$.

$(-2, 8) \xrightarrow{T_1} (1, 6) \xrightarrow{T_2} (-2, 8)$.

$(a, b) \xrightarrow{T_1} (a+3, b-2) \xrightarrow{T_2} (a, b)$.

$(x, y) \xrightarrow{T_2} (x-3, y+2) \xrightarrow{T_1} (x, y)$.

Therefore $T_1 \circ T_2 = i$.

6. (a) $T(a+c, b+d)$ (b) $T(-a, -b)$ (c) $T(0, 0)$.

7. Using the notation of Exercise 6,

$T(a, b) \circ T(c, d) = T(c+a, b+d)$

$T(c, d) \circ T(a, b) = T(a+c, d+b)$

Since $a+c = c+a$, and $b+d = d+b$ for all real numbers

$T(a, b) \circ T(c, d) = T(c, d) \circ T(a, b)$.

8. (a) $(2, 0) \xrightarrow{R_\ell} (-2, 0) \xrightarrow{R_m} (10, 0)$.

and $(2, 0) \xrightarrow{T(8, 0)} (10, 0)$.

(b) $(3, -4) \xrightarrow{R_\ell} (-3, -4) \xrightarrow{R_m} (11, -4)$

and $(3, -4) \xrightarrow{T(8, 0)} (11, -4)$.

(c) $(10, -3) \xrightarrow{R_\ell} (-10, -3) \xrightarrow{R_m} (18, -3)$

and $(10, -3) \xrightarrow{T(8,\ 0)} (18, -3)$.

9. Let $R_mR_\ell = F$. Then $(R_mR_\ell)(R_\ell R_m) = F(R_\ell R_m)$.

Using the associative property we may write

$$R_m(R_\ell R_\ell)\ R_m = F(R_\ell R_m).$$

But $R_\ell R_\ell = i$ since line reflections are involutions and $R_m i = R_m$. So

$$R_mR_m = F(R_\ell R_m)$$

$$i = F(R_\ell R_m).$$

Therefore F is the inverse of $R_\ell R_m$. Since the latter is not i, in general, then $F \neq R_\ell R_m$.

Proofs by coordinates are also available. The general result may also be stated as follows: If $R_mR_\ell = T(a, b)$ then $R_\ell R_m = T(-a, -b)$. This tells that the distances of the translations are the same, but they have opposite directions.

10. Compositions of transformations are associative. Since translations and line reflections are transformations, compositions of translation is associative and also composition of line reflections.

11. Exercise 6 answered this in terms of the notation T(a, b). In terms of coordinate rules we may say, if T_1 has rule $(x, y) \xrightarrow{T_1} (x+a, y+b)$ and T_2 has rule $(x, y) \xrightarrow{T_2} (x+c, y+d)$, then

$$(x, y) \xrightarrow{T_2 \circ T_1} (x+a+c, y+b+d)$$
$$(x, y) \xrightarrow{T_1 \circ T_2} (x+c+a, y+d+b)$$

12. The composition of two line reflections has been shown to be a translation if their axes are parallel. Since a translation is not a line reflection the answer to the question is no.

13. (a) The set of all translations in a plane and the operation of composition is a group because:
    - (i) The composition of two translations is a translation (See Exercise 11).
    - (ii) Compositions of translations (transformations) is associative. (See Exercise 10).
    - (iii) $i = T(0, 0)$ (notation of Exercise 6). Then $T(0, 0) \circ T(a, b) = T(a, b) \circ T(0, 0) = T(a, b)$.
    - (iv) $T(-a, -b)$ is the inverse of $T(a, b)$.

    (b) The set of line reflections with composition as operation is not a group since it is not an operational system. (See Exercise 12).

14. $R_\ell \circ R_\ell = T(0, 0)$. (See Exercise 12 Section 9.4).

9.7 Rotations and Half-Turns (4 days)

The first step in presenting (reviewing) rotations can be taken with the aid of an overhead projector. Using a grommet fasten a transparency to its base at a point near its center. Acetate sheets for the overhead projector are available in many colors at reasonable prices. Since even the writing with magic markers may be washed off. Larger sheets may be used over a white background without using the overhead projector. See problem 16 in Section 9.8.

Mark a set of points (see Figure 9.8) and the lines through them on the base, and on the transparency so that a point of the transparency coincides with a point of the base. Then rotate the transparency, through 30 degrees, say. This activity should clarify the nature of a rotation, and embark the student on explorations of rotations. Be careful to define positive and negative directions for rotations respectively, counter-clockwise and clockwise. Sooner or later students will confuse the measure of an angle with the measure of a rotation. An angle is not the same as a rotation. An angle is a set of points. A rotation is a transformation. The measures of angles vary from 0 to 180 (including 0 and 180, in some books, excluding them in others). The measure of rotations can be from -360 to +360, for our present purposes, and the set of all real numbers for other purposes.

The basic theorem in this section asserts that every rotation can be regarded as the composition of two line reflections whose axes intersect. A reversal of order of the line reflections results in a different rotation, in fact, the inverse rotation. Also of interest is the fact that there are an infinite number of pairs of line reflections that produce the same rotation. This leads to the conclusion that rotations are isometries and inherit properties of line reflections that survive a composition.

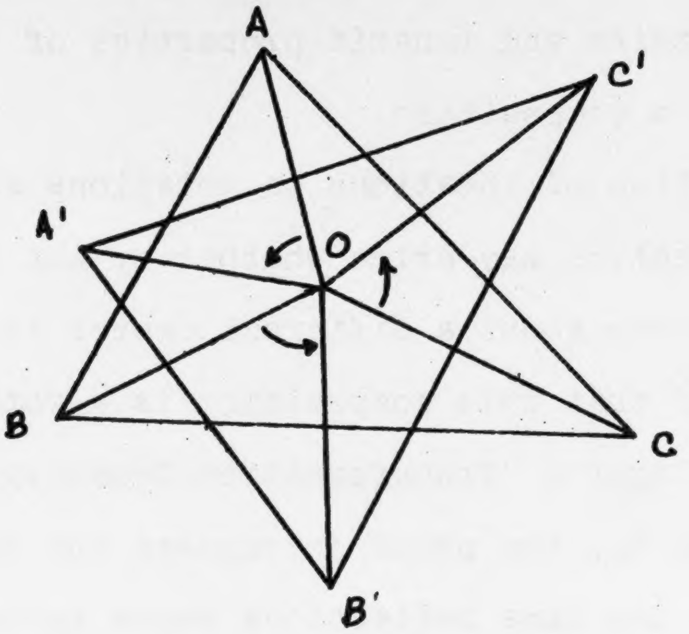
We restrict composition of rotations to rotations about the same center. The question may arise whether or not the composition of two rotations about a different center is also a rotation. A proof that this composition is a rotation is found on page 56, of Jeger's "Transformation Geometry." If the centers are O_1 and O_2, the proof decomposes the first rotation (about O_1) into two line reflections whose second axis is $\overleftrightarrow{O_1 O_2}$, and decomposes the second rotation (about O_2) into two line relections whose first axis is also $\overleftrightarrow{O_1 O_2}$. The center of the resultant rotation is the intersection of the remaining axes, and the measure of the resultant rotation is the sum of the measures of the component rotations.

We have treated half-turns as special cases of rotations where measures are 180 degrees. This implies that a point A_1, its image A', and the center O_1 of the rotation, are collinear with O, the midpoint of $\overline{AA'}$. For this reason a half-turn is also known as a symmetry or a reflection in a point.

The coordinate formula for a half-turn about (a, b) is important and should be remembered for exercises that follow. You may have to review the midpoint formula.

9.8 Exercise Solutions

1.



2. Since a rotation is the composition of two line reflections it has the properties that are conserved in the composition of two reflections. Therefore the images A', B', C' are collinear; B' is between A' and C'; AB = A'B', AC = A'C', BC = B'C'.

3. $r(P, 20) \circ r(P, 30) = r(P, 50)$

   If $A \xrightarrow{r(P,\ 20)} A_1 \xrightarrow{r(P,\ 30)} A'$, then

   $m\angle APA_1 = 20$, $m\angle A_1PH' = 30$, $\overrightarrow{PA_1}$ is between

   $\overrightarrow{PA}$ and $\overrightarrow{PA'}$; hence $m\angle APA' = 50$. Also

   $PA = PA_1 = PA'$ or $PA = PA'$

   Therefore $A \xrightarrow{r(P,\ 50)} A'$.

4. (a) r(P, 60) (b) r(Q, 10)

   (c) r(P, 170) (d) r(Q, 0)

5. Then inverse of $r(P, \theta)$ is $r(P, -\theta)$.

6. The system $(r(P, \theta), \circ)$ is an operational system for $r(P, \theta_2) \circ r(P, \theta_1) = r(p, \theta_1+\theta_2)$, is associative for the reason that it is a transformation: $r(P, 0)$ is the identity rotation, and the inverse of $r(P, \theta)$ is $r(P, -\theta)$ because their composition in either order is $r(P, 0)$.

7. (a) $(-3, 2)$ (b) $(2, -3)$ (c) $(0, 2)$ (d) $(-\sqrt{2}, \sqrt{3})$.

8. Using the rule $(x, y) \xrightarrow{\qquad} (2a - x, 2y - b)$ where (a, b) is the center of the half-turn. The images of A, B, C, D are respectively:

(a) $(2-3, -4+2) = (-1, -2)$

(b) $(2+2, -4-3) = (4, -7)$

(c) $(2-0, -4+2) = (2, -2)$

(d) $(2 - \sqrt{2}, -4 + \sqrt{3})$.

9. $(-2-3, 6+2) = (-5, 8)$

$(-2+2, 6-3) = (0, 3)$ $(-2 - \sqrt{2}, 6 + \sqrt{3})$

$(-2-0, 6+2) = (-2, 8)$

10. Taking the center of the half-turn as the origin of a rectangular coordinate system

$$(x, y) \xrightarrow{Ho} (-x, -y) \xrightarrow{Ho} (x, y).$$

Therefore $(x, y) \xrightarrow{Ho \circ Ho} (x, y)$ and H_O is an involution.

11. Let $P(a, b)$ and $Q(c, d)$ then:

$$A(x, y) \xrightarrow{H_P} A_1(2a-x, 2b-y) \xrightarrow{H_Q} A'(2c-2a+x, 2d-2b+y)$$

The rule of $H_Q H_P$ is that of a translation with rule

$$(x, y) \xrightarrow{\qquad} (x+2c-2a, y+2d-2b)$$

The slope from A to A' = $\frac{d-b}{c-a}$. From P to Q the slope is $\frac{d-b}{c-a}$. Therefore AA' || PQ and $\overrightarrow{AA'}$ and $\overrightarrow{PQ}$ have the same direction. Also AA' = $\sqrt{4(a-c)^2 + 4(b-d)^2}$ =

$$2\sqrt{(a-c)^2 + (b-d)^2} = 2PQ$$

12. Using data in Exercise 11

H_QH_P : $(x, y) \xrightarrow{\qquad} (x + 2c - 2a, y + 2d - 2b)$

on the other hand

H_PH_Q : $(x, y) \xrightarrow{\qquad} (x + 2a - 2c, y + 2b - 2d)$

$H_QH_P = H_PH_Q$ iff a = c and b = d so,

given $P \neq Q$ then $H_QH_P \neq H_PH_Q$

$$(x, y) \xrightarrow{H_P \circ H_Q} (x+2a-2c, y+2b-2d) \xrightarrow{H_QH_P} (x, y)$$

$\therefore$ (H_PH_Q) and (H_QH_P) are inverses.

13. Every line through the center of a half-turn is fixed, not pointwise. To prove this take the center of the origin and a line, y = mx. Since $(x, y) \xrightarrow{H_O} (-x, -y)$, $(x, mx) \xrightarrow{H_O} (-x, -mx)$. The coordinates (-x, -mx) satisfy y = mx. Hence y = mx is fixed under H_O, but not pointwise.

14. Let O be the origin of a rectangular coordinate system and ℓ with equation y = mx + b, $b \neq 0$, any line not containing O. Then $(x, mx+b) \xrightarrow{\qquad} (-x, -mx-b)$. The coordinates of the image satisfies y = mx-b, an equation of ℓ'. Therefore the slope of ℓ is equal to that of ℓ'. Hence $\ell \parallel \ell'$.

15. ABB'A' is a parallelogram because $\overline{AB'}$ and $\overline{A'B}$ having O

as midpoint bisect each other. For an alternate proof one may use coordinates and prove by the slope formula (assuming all lines have slopes) that opposite sides lie in parallel lines.

16. a. $(1, 0) \longrightarrow (a \cdot 1 + b \cdot 0, b \cdot 1 - a \cdot 0) = (a,b)$

b. $(a, b) \longrightarrow (a \cdot a + b \cdot b, b \cdot a - a \cdot b) =$

$(a^2 + b^2, 0)$, $(1, 0)$.

c. Any point P on ℓ has coordinates $(C, \frac{bc}{a+1}$

Then $(C, \frac{bc}{a+1}) \longrightarrow (ac + \frac{b^2c}{a+1}, bc - \frac{abc}{a+1})$. $ac + \frac{b^2c}{a+1} =$

$$\frac{a^2c + ac + b^2c}{a+1} = \frac{c(a^2+b^2) + ac}{a+1} = \frac{c+ac}{a+1} = c.$$

$$bc - \frac{abc}{a+1} = \frac{abc + bc - abc}{a+1} = \frac{bc}{a+1}.$$

Therefore, P is a fixed point and ℓ is a fixed line.

d. $(x, y) \xrightarrow{R_\ell} (ax + by, bx - ay)$.

Applying the transformation again,

$(ax + by, bx - ay) \longrightarrow (a(ax+by) + b(bx-ay), b(ax+by)$
$-a(bx-ay)) = (x, y)$.

17. If $A(x, y)$ is a fixed point then $x = ax-by$ and $y = bx+ay$. This system of equations has a unique solution $(x, y) = (0, 0)$ if $a \neq 1$. If $a = 1$ then $b = 0$ and the rule degenerates to $(x, y) \longrightarrow (x, y)$, the rule for the identity transformation, which may be considered a rotation through 0°. Hence this mapping, if it is not the identity, has only one fixed point, namely O (0, 0).

b. Let A have coordinates (c, d). Then A' has coordinates (ac-bd, bc+ad).

Calculating distances OA and OA',

$OA = \sqrt{c^2 + d^2}$

$O'A' = \sqrt{(ac-bd)^2 + (bc + ad)^2} = \sqrt{a^2c^2+b^2d^2+b^2c^2+a^2d^2}$

$= \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{c^2+d^2}$

Therefore OA = OA'.

c. $(1, 0) \longrightarrow (a\cdot 1-b\cdot 0, b\cdot 1+a\cdot 0) = (a, b)$

$(0, 1) \longrightarrow (a\cdot 0-b\cdot 1, b\cdot 0+a\cdot 1) = (-b, a)$

$(a, -b) \longrightarrow (a\cdot a-b(-b), b\cdot a+a(-b)) = (1, 0)$

18. Let $A(a, b)$ and $B(c, 0)$ be two points of ℓ.

Then $m = \frac{b}{a-c}$.

$A(a, b) \xrightarrow{Rx} A'(a,-b)$, $B(c, 0) \xrightarrow{Rx} B'(c, 0)$.

Slope of $A'B' = \frac{-b}{a-c} = - m$.

A similar proof shows that the slope of $\ell_2 = -m$.

19. Activity

20. From Exercise 16, $y = x$ is of the form $y = \frac{b}{a+1} x$ where $a = 0$ and $b = 1$.

The rule for R_ℓ is $(x, y) \longrightarrow (ax+by, bx-ay)$ which now becomes $(x, y) \longrightarrow (y, x)$.

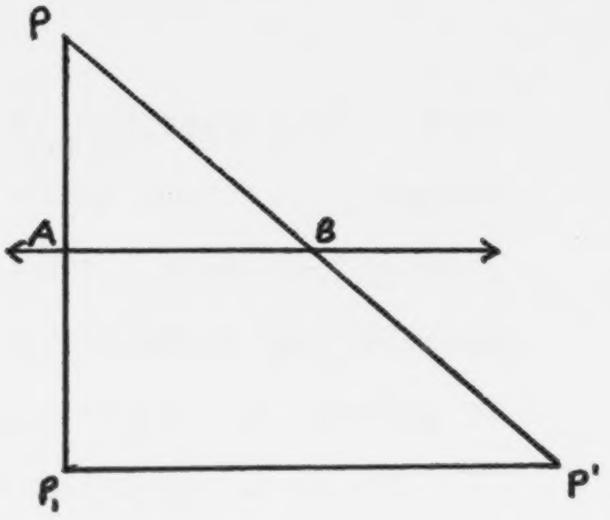
9.9 Composing Isometries. Glide Reflections. (2-3 days)

This section gives special attention to composition of isometries, arising out of our interest in regarding translations and rotations as compositions of line reflections. It leads naturally to the glide reflection, another type of isometry, and raises the question whether or not there are more. This question is answered in Section 9.11, where any isometry is shown to be the composition of no more than three line reflections. This section does more. It continues to prepare the student for an appreciation of the group structure of the set of isometries under composition.

It is interesting to note in passing that a translation is the composition of two half-turns about distinct centers. This fact permits us to regard a glide reflection as the composition of a line reflection and two half-turns, or a line reflection and four suitably chosen rotations.

9.10 Exercise Solutions

1. Let ℓ be the axis of the line reflection and also the x-axis of a rectangular coordinate system. Let T be the translation with rule $(x, y) \longrightarrow (x+a, -y)$. This T is in the direction of ℓ.
   $(x, y) \xrightarrow{T} (x+a, y) \xrightarrow{R_\ell} (x+a, -y)$, and
   $(x, y) \xrightarrow{R_\ell} (x, -y) \xrightarrow{T} (x+a, -y)$
   Therefore $R_\ell \circ T = T \circ R_\ell$.
2. Since the composition of two line reflections in parallel axes is a translation, the composition of three line reflections described is equal to the composition of a translation and a line reflection whose axis is parallel to the direction of the translation. By Exercise 1 this is a glide reflection. Also by Exercise 1, the answer to the last question is yes.



* 3. Let $F = T \circ R_\ell$ shere T is the translation component of F. Let $P \xrightarrow{R_\ell} P_1 \xrightarrow{T} P'$. Then ℓ bisects $\overline{PP_1}$ and is parallel to $\overleftrightarrow{P_1P'}$. Therefore ℓ bisects $\overline{PP'}$. (The line that bisects the side of a triangle and is parallel to a second side bisects the third side). (See Section 6.17 Chapter 6 problem 2).

4. Using the data in the solution to Exercise 3, let $\overleftrightarrow{PP_1}$ intersect ℓ in A and let $\overleftrightarrow{PP'}$ intersect ℓ in B.

Then $T = H_B \circ H_A$.

Thus $F = H_B \circ H_A \circ R_\ell$.

5. Let the glide reflection be $F = T \circ R_\ell$

Then $F \circ F = (T \circ R_\ell) \circ (T \circ R_\ell) = (T \circ R_\ell) \circ (R_\ell \circ T)$
(See Exercise 1)

$= T \circ (R_\ell \circ R_\ell) \circ T$ (The associative property of composition of mappings

$= T \circ i \circ T$ (line reflections are involutions)

$= T \circ T$ $(T \circ i = T)$

$= T'$ (The composition of two translations is a translation)

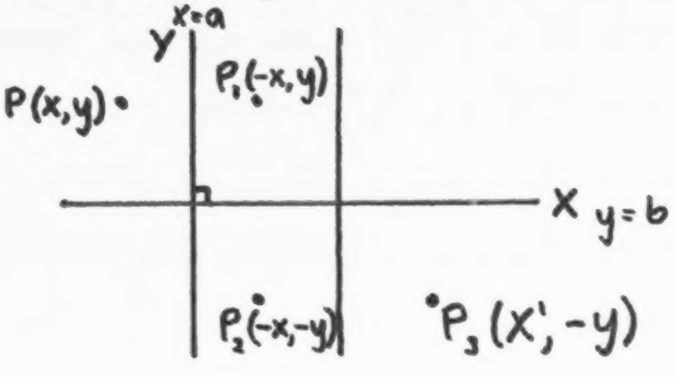
T' has the same direction as T, but twice the distance.

6. $A\ B\ A_1\ B_1$ is a parallelogram because its diagonals bisect each other. In general $A_1A_2B_2B_1$ is a parallelogram (it is not a parallelogram if A_2 is collinear with B_1, A_1 but $A_1B_1 = A_2B_2$ because translations are isometries) because $A_1B_1 = A_2B_2$ and $\overline{A_1B_1}$ and $\overline{A_2B_2}$ lie in parallel lines. It follows that $AB = B_2A_2$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{B_2A_2}$. Hence ABA_2B_2 is a parallelogram. Therefore the isometry that maps A onto A_2 and B onto B_2 is a half-turn, because the diagonals $\overline{AA_2}$ and $\overline{BB_2}$ bisect each other. (It is assumed that A_2 is not collinear with A, B.) An occupation with the collinear cases requires the following: If O is on $\overleftrightarrow{AB}$ it still holds since a half-turn is an isometry and $AB = A_1B_1$ and $A_1A_2B_2B_1$ is a parallelogram by the arguement above.

7. Let T be the translation and H_O the half-turn. Let A and B be two points. Let $A \xrightarrow{T} A_1$, $B \xrightarrow{T} B_1$. Then ABB_1A is a parallelogram and $\overleftrightarrow{AB} \parallel \overleftrightarrow{A_1B_1}$ and $AB=A_1B_1$. Let O be non-collinear with A_1 and B_1, then $A_1 \xrightarrow{H_O} A_2$ and $B_1 \xrightarrow{H_O} B_2$. Again $A_1B_1A_2B_2$ is a parallelogram, and $\overleftrightarrow{A_1B_1} \parallel \overleftrightarrow{A_2B_2}$, $A_1B_1 = A_2B_2$. Therefore $\overleftrightarrow{AB} \parallel \overleftrightarrow{B_2A_2}$ and $AB = B_2A_2$. Therefore ABA_2B_2 is a parallelogram and its diagonals bisect each other. Thus, under the half-turn in midpoint of $\overline{AA_2}$ (or $\overline{BB_2}$) $A \longrightarrow A_2$ and $B \longrightarrow B_2$. Thus $H_O \circ T$ is a half-turn. If O is collinear with A_1 and B_1 we need the statement cited in the last sentence of the solution of Exercise 6.

8. We need to consider only x-coordinates of P and the sequences of images. If x_1 is the coordinate of P then the coordinate of P_1, its reflection in the y-axis, is $-x$ since $OP = OP_1$. If P_2 is the reflection of P_1 in line a (with equation $x=a$) then its coordinate is x' such that $a = \frac{1}{2}(x' - x_1)$. Solving gives $x' = 2b - 2a + x_1$. The reflection of P_2 in b (the line with equation $x=b$) is P_3. Its coordinate x'' is such that $b = \frac{1}{2}(x' - x'')$ or $2b = 2a + x_1 - x''$. Solving gives $x'' = 2a - 2b + x_1$. Consider the distance PP_3. $PP_3 = |2a - 2b + x_1 - x_1|$ $= 2|b-a|$ or $\frac{1}{2}PP_3 = |b-a|$. There is a fixed line where the equation is $x = a - b + x_1$, and P's distance to this line is $|b-a|$, while P_3's distance to it is also $|b-a|$. Thus this line is the perpendicular bisector of $\overline{PP_3}$. We conclude that composition of the three line reflections is a line reflection.

9. Take a to be the y-axis and b the x-axis of a rectangular coordinate system and then c has equation $x=a$.



$R_c \circ R_b \circ R_a$

$x' - c = c + x$

P_3 $x' = 2c + x$

P_3 is the image of a point P_4 under R_b.

This point must be P_4 $(2c + x, y)$

P_4 is the image of a translation whose direction is $||$ to x-axis and distance $= 2c$.

10. (a) We must show that the slope of ℓ is the same as the slope of the line of direction. The slope of ℓ is $\frac{b}{a+1}$ and the direction line's equation has a slope of $\frac{bc}{(a+1)c} = \frac{b}{c}$.

(b) We can represent any point of ℓ as having coordinates $((a+1)d, bd)$ where d can take on all values, since these coordinates satisfy the equation of ℓ. If we apply the coordinate rule on these coordinates

$$(x, y) \longrightarrow (ax+by+(a+1)c,\ bx-ay+bc).$$

Considering first the effect on $(a+1)d$, then on bd we get $(a+1)d \longrightarrow a(a+1)d+b(bd)+(a+1)c = a^2d+ad+b^2d+ac+c$

$$= d(a^2+b^2)+ad+ac+c = d+ad+ac+c$$

$$bd \longrightarrow b(a+1)d-a\ bd+bc = bd+bc.$$

If the "image" values satisfy the equation of the line ℓ the proof is complete. Use the equivalent equation $(a+1)y = bx$. Then $(a+1)(bd+bc)$ should be equal to $b(d+ad+ac+c)$, and they are.

The rule should assign a point of ℓ to a point of ℓ because under a reflection in a line the line itself is fixed and only the translation "moves" that point.

(c) $(x, y) \xrightarrow{T} (x+(a+1)c,\ y+bc)$

$(x, y) \xrightarrow{R_\ell} (ax+by,\ bx-by)$ See section 9.8 problem 16.

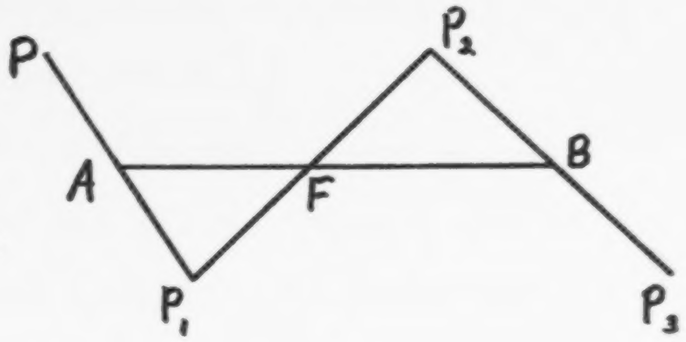
$(x, y) \xrightarrow{T \circ R_\ell} (ax+by+(a+1)c,\ bx-by+bc)$

11. If X is the midpoint of $\overline{PP_3}$ then $\overleftrightarrow{XC} \,||\, \overleftrightarrow{PP_2} \,||\, \overleftrightarrow{AB}$ and $\overleftrightarrow{XA} \,||\, \overleftrightarrow{P_3P_1} \,||\, \overleftrightarrow{BC}$ because if a line joins the midpoint of 2 sides of a triangle it is $||$ to the third side. Thus ABCX is a parallelogram. But there is only one point that can serve as X. Therefore for all P there is one point that is the midpoint of $\overline{PP_3}$.

This shows that the composition of H_A, H_B, H_C which maps P onto P_3 is the same as the H_D which also maps P onto P_3. Hence the composition of three half-turns is a half-turn.

This conclusion is not invalidated if A, B and C are collinear.

12. This is proved if we can show that F is the midpoint of $\overline{PP_3}$. This can be done by showing that $PP_1P_3P_2$ is a parallelogram for then F is the midpoint of PP_3. A coordinate proof follows on taking $\overleftrightarrow{AB}$ as the x-axis of a rectangular coordinate system with F as origin. Then A and B may be assigned coordinates (-a, 0) and (a, 0).



$P(x, y) \xrightarrow{H_A} P_1(-2a-x, -y) \xrightarrow{H_F} (2a+x, y) \xrightarrow{H_B}$ $P_3(-x, -y)$ and finally $P_3(-x, -y) \xrightarrow{H_F} P(x, y)$.

9.11 The Three Line Reflection Theorem (2-3 days)

We have not shown a figure for the first theorem of this section. We hope that students will understand it without a figure. This they can do if they see, for instance, that from A'X' = A'X" we conclude that A' is equidistant from X' and X". To see a geometric proof by noting the symbols used to represent geometric entities is a valuable experience in that it brings home the idea that our system is an abstract one, and that the figure is a model.

Before teaching the second theorem encourage students to examine Figure 9.11e, in particular to trace the paths from A to A'; from B to B_1 to B'; and from C to C_1 to C_2 to C'. These are indicated by arrows along dotted lines. As they trace these paths ask them to tell the distances that are preserved in their "motion". If they can do this they may be able to follow the proof without looking at the figure.

9.12. This exercise should be done in class by tracing a cardboard triangle in two positions and finding 3 lines of reflection that will map one on the other. Often this can be achieved in more than one way. Students can repeat this with two congruent triangles placed on a ditto. Special cases requiring one or two reflections in a line may be examined.

9.13 Directed Isometries (1-2 days)

We have not said much about orientation along a line or along parallel lines. This concept can be made precise in terms of parallel rays or anti-parallel rays. The former are two rays on two parallel lines that are in the same half-plane of the line containing their vertices, or two rays on a line whose intersection is a ray. Two anti-parallel rays are rays in parallel lines that are in opposite half-planes of the line containing their vertices, or two rays on a line where intersection is not a ray. Two rays have the same sense if they are parallel; they have opposite senses if they are anti-parallel.

The above concepts can be presented informally, if you wish.

It is easier to present informally the concept of orientation of a plane. Though the concept is simple, when presented informally, its application can be quite profound. Note for instance, the simple proofs that can be given in Exercises 6 and 7 of Section 9.14.

9.14 Exercise Solutions

Note to teacher: All exercises should be done and completed in order. One builds on another. Exercise 10 should be starred, or else students should be given hints for the proof.

1. A line reflection reverses the sense of three noncollinear

points. A second line reflection restores the sense. A third reverses the sense. Hence the composition of three line reflections is an opposite isometry.

2. A pair of line reflections is a direct isometry. An even number of line reflections is a composition of pairs. The composition of direct isometries is direct since no reversal of sense takes place for each pair. An odd number of compositions of line reflections may be regarded as the composition of an even number of reflections and one extra reflection (2n+1 = (2n)+1). The composition of the even number of reflections is direct, while the last reflection reverses sense. Hence the composition of odd number of line reflections is an opposite isometry.

3. A glide-reflection is the composition of a translation and a reflection in a line. The first preserves sense; the second reverses it. The composition therefore reverses sense and is an opposite isometry.

4. a. A translation preserves sense. The composition of any number of translations continues to preserve sense. Hence the composition of any number of translations is a direct isometry.

   b. The same argument applies to rotations as to translations, since rotations are direct isometries. The composition of any number of Rotations is a direct isometry.

c. The same argument applies to half-turns as to rotations, so the composition of any number of half-turns is a direct isometry.

5. a. Any permutation of (reverse, reverse, preserve) results in "preserve." Hence the composition of the 3, in any order, is a direct isometry.

   b. A half-turn preserves sense. To produce an opposite isometry, the half-turn should be composed with an opposite isometry. The compositions of an even number of line reflections is a direct isometry while the composition of an odd number is an opposite isometry. Hence we should use an odd number of line reflections to produce an opposite isometry.

6. Since a half-turn is a direct isometry and the composition of any number of half-turns continues to be direct, and since a line reflection is an opposite isometry, it follows that the composition of any number of half-turns cannot be a line reflection.

7. The same argument applies to rotations as the one in Exercise 6 for half-turns. The fact that a rotation may be the identity transformation does not invalidate the conclusion.

8. a. Since a glide reflection is an opposite isometry, we need another opposite isometry with which the glide reflection should be composed to produce a direct isometry. This can be a line reflection

or another glide-reflection.

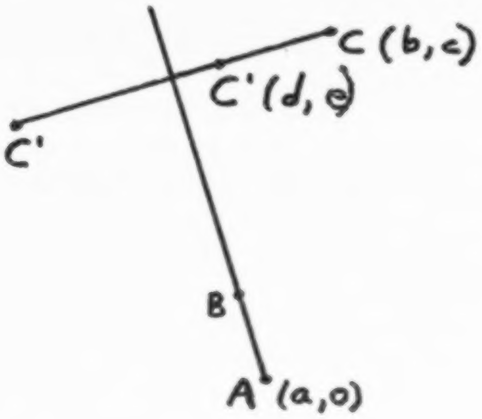
b. To produce an opposite isometry we need to compose the glide-reflection with a direct isometry, and this can be a rotation (including a half-turn), or a translation.

9. a. The identity mapping preserves sense; hence it is a direct isometry.

b. If an isometry is direct, its inverse must continue to preserve sense; that is, its inverse must also be direct, for the composition to be direct. (The composition must be direct, since the identity mapping is direct).

If an isometry is opposite its inverse must reverse the reversal of sense to restore the sense, so that the composition is the identity, which is direct. Hence its inverse must also be opposite.

10. Let the two fixed points be A and B, and let C be a third point not in $\overleftrightarrow{AB}$. Let the image of C be C' under the isometry. If C and C' are on opposite sides of $\overleftrightarrow{AB}$ then the sense in (A, B, C) is reversed in (A', B', C'). This denies the hypothesis that the isometry is direct. Therefore C and C' are on the same side of $\overleftrightarrow{AB}$. Because the isometry preserves distance AC = AC', BC = BC' we can show that C = C' by taking



$\overleftrightarrow{AB}$ as the x-axis of a rectangular coordinate system, with B as origin. Let A have coordinates $(a, 0)$, $C(b, c)$, $C'(d, e)$. Then using $(AC)^2 = (AC')^2$ we get $(a-b)^2+c^2 = (a-d)^2 + e^2$, and from $(BC)^2 = (BC')^2$, $b^2+c^2 = d^2+e^2$. These imply $b^2 = d^2$ and therefore $b = d$. Hence $c^2 = e^2 \Rightarrow |c| = |e|$. These imply $b = d$. Since C and C' are on the same side of the axis $\overleftrightarrow{AB}$ we can then show $c = e$. Hence $C = C'$. By the lemma of Section 9.11 an isometry is uniquely determined by its effect on three noncollinear points. Since the identity transformation leaves three noncollinear points fixed, the isometry is the identity.

9.15 Groups of Isometries (2 days)

We have not considered the group of all rotations in a plane, restricting our attention to rotations about one point. We omitted a consideration of the larger group to avoid the difficulty of showing that the composition of two rotations $r_1(A_1, \theta_1)$ and $r_2(A_2, \theta_2)$ is $r(Q, \theta_1 + \theta_2)$, where Q is a point in the plane. (See Jeger, page 56). If you wish to develop the larger group, see Problem 3 in the Suggested Test Items of this chapter.

You will find an example of a finite group of isometries, the group that leaves a figure invariant, in Exercise 9 of Section 9.16.

9.16 Exercise Solutions

Note to teacher: Problem 9 is important; it will be used in the following set of exercises. Also, add to Exercise 1, part b, "Using your results in part a, show that the set of translations (in a plane) is a group.

1. a. (i) $T_{(A,\ A)}$ is the identity transformation;

   (ii) $T_{(B,\ A)}$ is the inverse of $T_{(A,\ B)}$; $T_{(B,\ C)} \circ T_{(A,\ B)} = T_{(A,\ C)}$

   (iii) $T_{(B,\ C)} \circ T_{(A,\ B)} = T_{(A,\ C)}$.

   b. The answer in a demonstrates that the identity transformation may be regarded as a translation; and every translation has an inverse that is also a translation. The associative property has been demonstrated for all isometries and therefore belongs to any subset of the set of isometries. We conclude that the set of translations (in a plane) is a group.

2. The set of direct isometries consist of all translations and half-turns. (1) Since the composition of direct isometries is a direct isometry, the set is closed. Since the direct isometry is unique, the set of direct isometries is an operational system. (2) The identity transformation is a special case of a transformation or the composition of a half-turn with itself. (3) We have seen in Exercise 1 that every translation has

an inverse in the set. If H is a half-turn the $H \circ H = 1$. Therefore H is its own inverse. (4) We have established that the associative property belongs to the set. Therefore the set of direct isometries is a group.

3. No; the set of opposite isometries is not a group because the composition of two such isometries is direct and hence not in the set (the set is not closed). In fact this shows that the set of opposite isometries is not an operational system.

4. No; the composition of two half-turns is a translation, which is not in the set of half-turns. Therefore this set is not even an operational system. Also, there is no identity, yet it contains its own inverses.

5. Yes. See Exercise 2 above; and by the definition of a sub-group.

6. Let P be the point about which the rotations take place and let $a > 0$. (a represents the number of degrees of rotation. It may be any number. It is not to be confused with the measure of an angle, which is a positive number less than 180).

(1) Since $r(P, b) \circ r(P, a) = r(P, a+b)$ is a rotation, we have here an operational system.

(2) Furthermore $r(P, 0) = 1$ (identity requirement)

(3) For each $r(P, a)$, $r(P, a) \circ (P, -a) = r(P, 0)$

(inverse requirement)

(4) The associative property requirement is satisfied

because composition of mappings is associative in general. Hence the set of rotations with the same center form a sub-group.

7. Since f and g are isometries, each has an inverse, f^{-1} and g^{-1}. Since the set of isometries is a group we can start the proof with $f \circ g \circ g^{-1} \circ f^{-1} = f \circ (g \circ g^{-1}) \circ f^{-1}$ $= f \circ i \circ f^{-1} = f \circ f^{-1} = i$

   operating on the left with $(f \circ g)$ we get

   $$(f \circ g)^{-1} \circ (f \circ g) \circ (g^{-1} \circ f^{-1}) = (f \circ g)^{-1} \circ i$$

   $$i \circ (g^{-1} \circ f^{-1}) = (f \circ g)^{-1}$$

   and finally $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$

8. $(f_1 o f_2 o f_3 \ldots o f_n)^{-1} = f_n^{-1} \circ f_{n-1}^{-1} \ldots f_2^{-1} \circ f_1^{-1}$.

9. (a) $A \xrightarrow{r_2} C$ $\qquad \overline{AB} \xrightarrow{r_2} \overline{CA}$

   $B \xrightarrow{r_2} A$ $\qquad \overline{BC} \xrightarrow{r_2} \overline{AB}$

   $C \xrightarrow{r_2} B$ $\qquad \overline{CA} \xrightarrow{r_2} \overline{BC}$

   In short, $\Delta ABC \xrightarrow{r_2} \Delta CAB$

   (b) $A \xrightarrow{R_1} A$ $\qquad \overline{AB} \xrightarrow{R_1} \overline{CA}$

   $B \xrightarrow{R_1} C$ $\qquad \overline{BC} \xrightarrow{R_1} \overline{BC}$

   $C \xrightarrow{R_1} B$ $\qquad \overline{CA} \xrightarrow{R_1} \overline{AB}$

   In short, $\Delta ABC \xrightarrow{R_1} \Delta ACB$

   (c) $A \xrightarrow{R_2} C$ $\qquad \overline{AB} \xrightarrow{R_2} \overline{BC}$

   $B \xrightarrow{R_2} B$ $\qquad \overline{BC} \xrightarrow{R_2} \overline{AB}$

   $C \xrightarrow{R_2} A$ $\qquad \overline{CA} \xrightarrow{R_2} \overline{CA}$

   In short, $\Delta ABC \xrightarrow{R_2} \Delta CBA$

(d) $A \xrightarrow{R_3} B$ $\qquad$ $\overline{AB} \xrightarrow{R_3} \overline{AB}$

$B \xrightarrow{R_3} A$ $\qquad$ $\overline{BC} \xrightarrow{R_3} \overline{CA}$

$C \xrightarrow{R_3} C$ $\qquad$ $\overline{CA} \xrightarrow{R_3} \overline{BC}$

In short, $\Delta ABC \xrightarrow{R_3} \Delta BAC$

(e)

o	i	r_1	r_2	R_1	R_2	R_3
i	i	r_1	r_2	R_1	R_2	R_3
r_1	r_1	r_2	i	R_2	R_3	R_1
r_2	r_2	i	r_1	R_3	R_1	R_2
R_1	R_1	R_3	R_2	i	r_2	r_1
R_2	R_2	R_1	R_3	r_1	i	r_2
R_3	R_3	R_2	R_1	r_2	r_1	i

(f) (S, o) is a group because it is an operational system, is associative; has i as identity element in S; each row and column has an i entry- hence each element has an inverse in S.

(g) Subgroups, in addition to (S, o) are $(\{i_1, r_1, r_2\}, o)$, $(\{i_1, R_1\}, o)$, $(\{i_1, R_2\}, o)$, $(\{i_1, R_3\}, o)$, $(\{i\}, o)$.

9.17 Isometry, Congruence, and Symmetry (4 days)

In this section we use the concept of an isometry to shed light on the nature of a congruence and a symmetry. A congruence is a relation between two figures. A symmetry is a property of a figure. It is a triumph of the concept of isometry that it is able to define both a congruence and a symmetry with simplicity when these concepts are otherwise quite difficult to define.

You should note that we do not develop any theorems such as the SAS, ASA and SSS principles for triangles. We are concerned here only with the definition of a congruence between figures, and this definition serves as the working program to prove any kind of figures congruent.

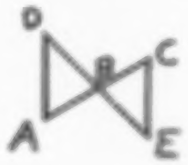
9.18 Exercise Solutions

Note to teacher: Exercise 7 is important.

1. Using B as the center of a half-turn,

   $A \longrightarrow C$, $B \longrightarrow B$, and $D \longrightarrow E$

   Therefore $\triangle ABD \longrightarrow \triangle CBE$ and $\triangle ABD \cong \triangle CBE$.



2. Using the reflection in $\overleftrightarrow{CD}$,

   a. $A \longrightarrow B$, $C \longrightarrow C$, $D \longrightarrow D$.

   Hence $\triangle ACD \longrightarrow \triangle BCD$, and $\triangle ACD \cong \triangle BCD$.

   b. $\angle CAD \longrightarrow \angle CBD$. Therefore $m\angle CAD = m\angle CBD$

   since isometries preserve angle measures.

   c. $\overline{AC} \longrightarrow \overline{BC}$, $AC = BC$. Isometries preserve distance.

   d. $A \longrightarrow B$, $C \longrightarrow C$

   $M \longrightarrow M$. $\therefore \triangle ACM \longrightarrow \triangle BCM$ and $\triangle ACM \cong \triangle BCM$.

   Since $\angle CAB \longrightarrow \angle CBA$, $m\angle CAB = m\angle CBA$.

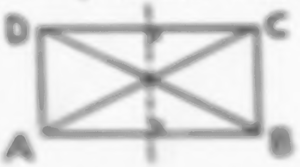
e. The reflection in $\overleftrightarrow{CD}$ maps ADBC onto BDAC. Hence ADBC is symmetric, with line of symmetry $\overleftrightarrow{CD}$.

3. No, the proof would be the same since $\overleftrightarrow{CD}$ will still be the perpendicular bisector of $\overline{AB}$ and reflection in $\overleftrightarrow{CD}$ will map $\Delta ACD \longrightarrow \Delta BCD$ and hence $\Delta ACD \cong \Delta BCD$.

4. Any two circles having radii of same length are congruent. One way to show this is to use the midpoint of the segment joining their centers as the center of a half-turn. This half-turn maps one circle onto the other. Other isometries are (a) The translation that maps one center onto the other, (b) the line reflection in the perpendicular bisector of the segment joining the centers of the circles.

A drawing showing any one of these isometries is satisfactory.

5. a. Under the line reflection in the perpendicular bisector of $\overline{AB}$, $A \longrightarrow B$, $B \longrightarrow A$ and $D \longrightarrow C$. Hence $\Delta ABD \longrightarrow \Delta BAC$ and $\Delta ABD \cong \Delta BAC$.



b. ABCD is symmetric under (1) The line reflections in the perpendicular bisector of $\overline{AB}$, or $\overline{DC}$ (2) The perpendicular bisector of $\overline{DA}$ or $\overline{BC}$. (3) The half-turn in the intersection of $\overline{AC}$ and $\overline{BD}$.

6. The first congruence uses the rotation $r_0(O, 120)$. The

The second use $r_2(0, 240)$ or $r_2(0, -120)$. Assume the discussion in Section 9.16 Ex. 9 that shows that r_1 and r_2 leave the triangle invariant.

Recall that by the rotation $r_1(0, 120)$

$$A \xrightarrow{r_1} B, \quad B \xrightarrow{r_1} C, \quad 0 \xrightarrow{r_1} 0$$

Hence $\Delta AOB \xrightarrow{r_1} \Delta BOC$ and $\therefore \Delta AOB \cong \Delta BOC$

Similarly, by $r_2(0, 240)$,

$$A \xrightarrow{r_2} C, \quad B \xrightarrow{r_2} A, \quad 0 \xrightarrow{r_2} 0$$

Hence $\Delta AOB \xrightarrow{r_2} \Delta COA$ and $\therefore \Delta AOB \cong \Delta COA$.

7. a. The identity transformation is an isometry. Therefore any figure is congruent to itself.

   b. Let I be the isometry that maps F onto F'. Then the inverse of I exists and is an isometry. (The set of isometries is a group) it maps F' onto F. Hence $F' \cong F$.

   c. Let I_1 and I_2 be the isometries that map F_1 onto F_2 and F_2 onto F_3, respectively. The composition $I_2 \circ I_1$, maps F_1 onto F_3. Hence $F_1 \cong F_3$.

   Since the congruence relation is reflexive, symmetric, and transitive it is an equivalence relation.

8. a. Five: 2 rotations, 3 line reflections (see Exercise 9, Section 9.16).

   b. One: line reflection (the line is the perpendicular bisector of the base).

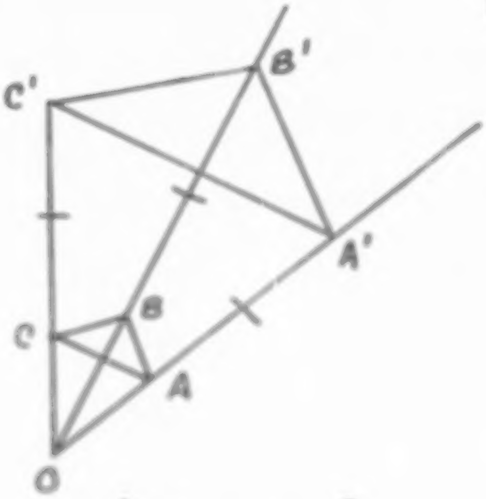
   c. Infinite: a rotation of any amount about the center; a line reflection in any diameter.

9.19 Other Transformations. Dilations and Similarities. (3 days)

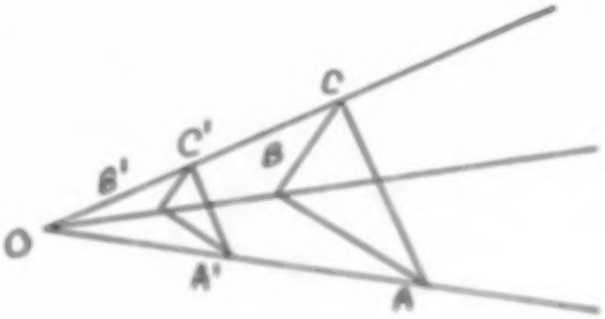
We use the term "dilation" in the same sense as Jeger uses "enlargement" and others use "homothety". As we use the term, a "similarity" is the composition of a dilation and an isometry. There is an unhappy variation in the literature in the use of the terms. If students are asked to do any reading, aside from the text book, they should be apprised of this variation.

9.20 Exercise Solutions

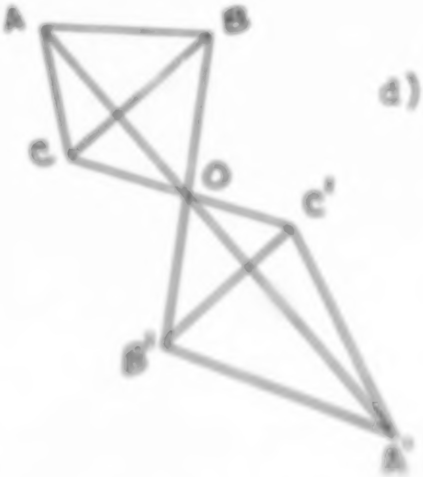
1. a)



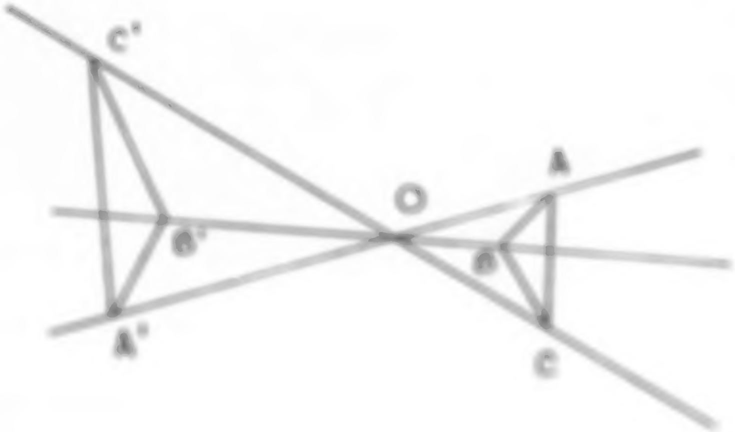
b)



c)



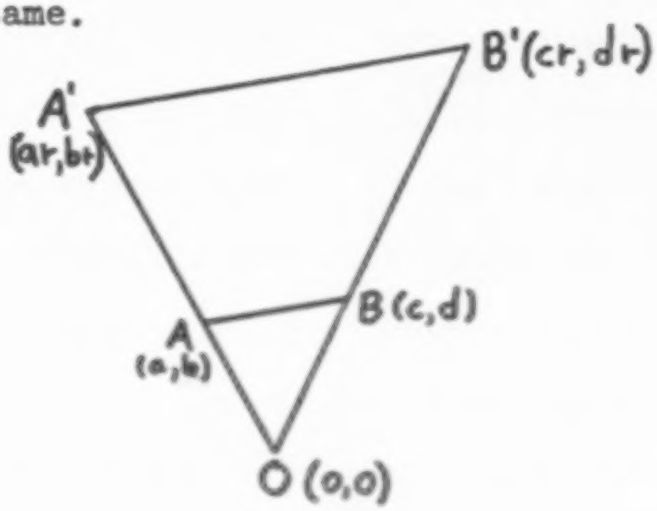
d)



There are other possible drawings.

2. The rules of assignment for a dilation about a center O with scale factor -1, and for a half-turn about the same center produce exactly the same image for each point. Hence such a dilation and half-turn are the same.

3.



Making such a drawing a measurement of $\overline{AB}$ and $\overline{A'B'}$ should be sufficient to satisfy the demands of the Exercise. If a general proof is required we can use a rectangular coordinate system which has the center of the dilation as origin, that assigns coordinates (a, b) to A and (c, d) to B. If the scale factor of the dilation is r then A' has coordinates (ar, br) and B' has coordinates (cr, dr). The calculations for distances AB and A'B' should prove A'B' = r(AB).

$$
\begin{aligned}
AB &= \sqrt{(a-c)^2 + (b-d)^2} \\
A'B' &= \sqrt{(ar-cr)^2 + (br-dr)^2} \\
&= r\sqrt{(a-c)^2 + (b-d)^2} \\
&= r\,AB.
\end{aligned}
$$

4. Let the collinear points be A, B, C with B between A and C. Then AB + BC = AB. Let r be the scale factor for the dilation and A', B', C' the images of A, B, C. Then, by Exercise 3 above

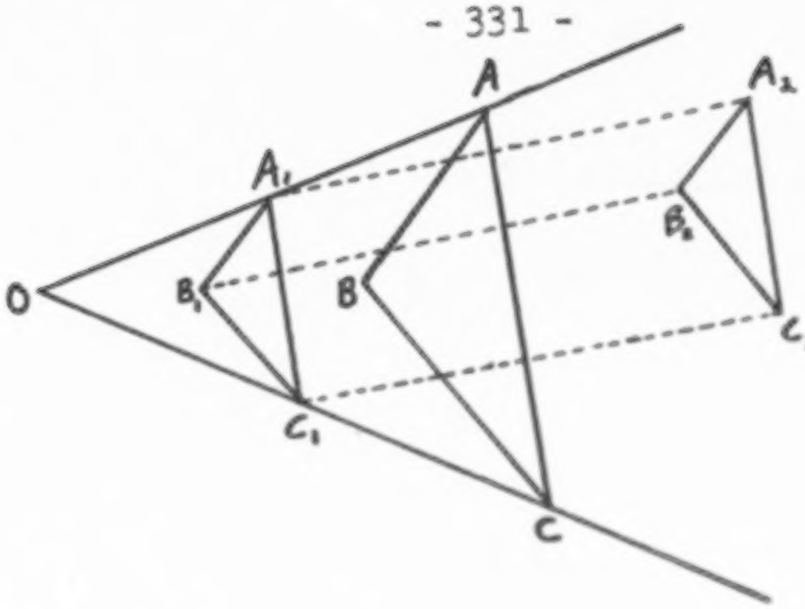
$$A'B' = rAB, \; B'C' = rBC, \; A'C' = rAC.$$

$AB+BC = AC$ implies $r(AB) + r(BC) = r(AC)$ or $A'B'+B'C' = A'C'$. Therefore B' is between A' and C'. This proves not only that A', B', C' are collinear, but also that the betweeness relation is preserved by a homothety, that dilations preserve lines, rays, segments.

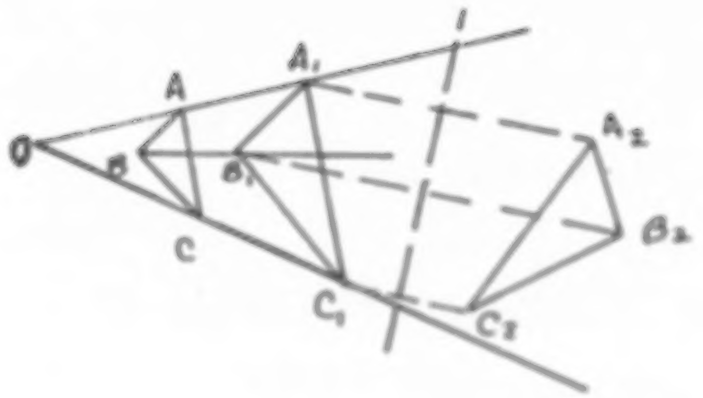
5. If dilation h_1 of O has scale factor r_1 and dilation h_2 of O has scale factor r_2 then $h_2 \circ h_1$ is a dilation of O with scale factor $r_1 r_2$. Thus the system is an operational system. (Associativity) Composition of dilations is associative since dilations are transformations. (Identity) The dilation of O with scale factor 1 assigns each point to itself. Hence it is the identity transformation. (Inverse) The composition of h_1 of O with scale factor a with h_2 of O with scale factor $\frac{1}{a}$ is a dilation of O with scale factor 1. Since $a \neq 0$ every dilation of O has an inverse which is a dilation of O.

This completes the proof.

6.



7.

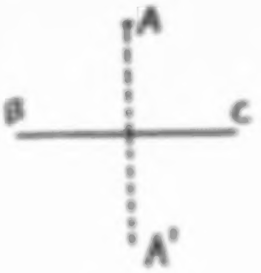


8. A similarity is direct if it preserves the sense of three noncollinear points. It is opposite if it reserves the sense of three noncollinear points.

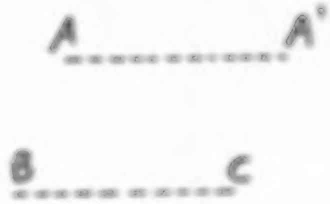
9. Since a similarity is the composition of a dilation and an isometry it is sufficient to know that dilations preserve angle measures to conclude that similarities preserve them also. One can prove that dilations preserve angle measures by first proving that the image of a line under a dilation is parallel to the line. This leads to the conclusion that the image of an angle under a homothety has the same measure as that or the original.

9.22 Summary Exercise Solutions (2 days)

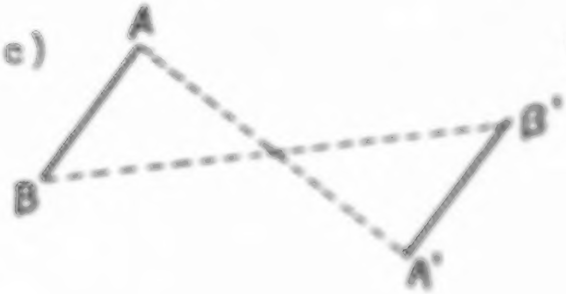
1. a)



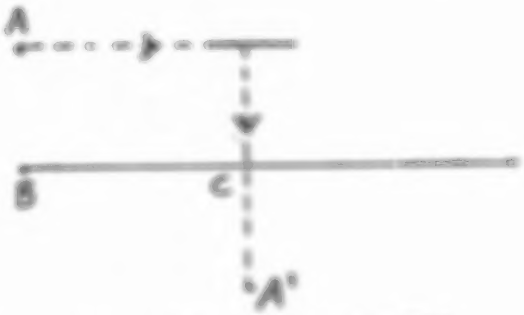
b)



c)



d)



2. a. Under the half-turn about the midpoint of $\overline{AC}$
$A \longrightarrow C$, $C \longrightarrow A$, $B \longrightarrow D$, $D \longrightarrow B$. Therefore
$\overline{AB} \longrightarrow \overline{CD}$, $\overline{BC} \longrightarrow \overline{DA}$, $\overline{CD} \longrightarrow \overline{AB}$ and $\overline{DA} \longrightarrow \overline{BC}$,
or $ABCD \longrightarrow CDAB$ and $ABCD \cong CDAB$.

b. No, because B is not mapped onto A, nor any other vertex of ABCD.

c. No, unless ABCD is a rectangle.

3. a. Each vertex is amppedoonto an adjacent vertex because $\overline{AC} \perp \overline{BD}$ and $\overline{AC}$ and $\overline{BD}$ bisect each other.

$A \xrightarrow{r_A} B$

$B \xrightarrow{r_A} C$

$C \xrightarrow{r_A} D$

$D \xrightarrow{r_A} A$

In short $ABCD \xrightarrow{r_A} BCDA$, so $ABCD \cong BCDA$.

b. $ABCD \xrightarrow{r_A} CDAB$ so $ABCD \cong CDAB$

c. $ABCD \xrightarrow{r_A} DABC$ so $ABCD \cong DABC$

d. The axis is the perpendicular bisector of $\overline{AB}$ and $\overline{CD}$. Therefore, under the reflection, $A \longrightarrow B$, $B \longrightarrow A$, $C \longrightarrow D$, $D \longrightarrow C$ and $ABCD \longrightarrow BACD$ so $ABCD \cong BADC$.

e. Since $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$ $A \longrightarrow A$, $B \longrightarrow D$, $C \longrightarrow C$, $D \longrightarrow B$, and $ABCD \longrightarrow ADCB$ so $ABCD \cong ADCB$.

f. (1) identity = i (2) A reflection in the line through the midpoints of $\overline{BC}$ and $\overline{DA}$ (R_2) (3) A reflection is $\overline{BD}$ (R_4).

g.

	i	r_1	r_2	r_3	R_1	R_2	R_3	R_4
i	i	r_1	r_2	r_3	R_1	R_2	R_3	R_4
r_1	r_1	r_2	r_3	i	R_3	R_4	R_1	R_2
r_2	r_2	r_3	i	r_1	R_2	R_1	R_4	R_3
r_3	r_3	i	r_1	r_2	R_3	R_4	R_2	R_1
R_1	R_1	R_3	R_2	R_4	i	r_2	r_1	r_3
R_2	R_2	R_4	R_1	R_3	r_2	i	r_3	r_1
R_3	R_3	R_2	R_4	R_1	r_3	r_1	i	r_2
R_4	R_4	R_1	R_3	R_2	r_1	r_3	r_2	i

h. $\{i, r_1, r_2, r_3\}$.

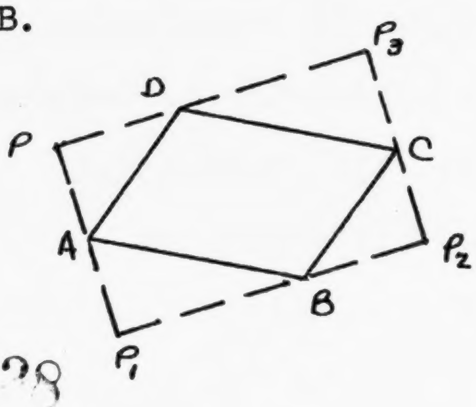
i. $\{i, R_1\}$, $\{i, R_2\}$, $\{i, R_3\}$, $\{i, R_4\}$, $\{i, r_2\}$.

4. Let P be any point not in AB.

Then $P \xrightarrow{H_A} P_1$

$P_1 \xrightarrow{H_B} P_2$

Thus $P \longrightarrow P_2$

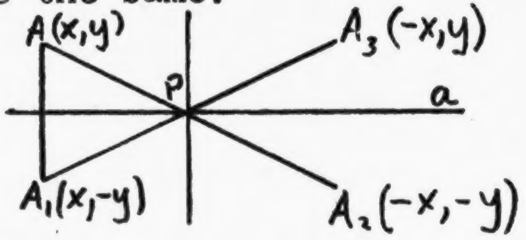


Now to find the image of P under $H_C \circ H_D$.

$$P \xrightarrow{H_D} P_3$$

If $P_3 \xrightarrow{H_C} P_2$ then the statement is proved.

From $P \xrightarrow{H_B H_A} P_2$ it follows $PP_2 = 2AB$ and $\overleftrightarrow{PP_2} \parallel \overleftrightarrow{AB}$ But $AB = DC$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$. Therefore $PP_2 = 2DC$ and $\overleftrightarrow{PP_2} \parallel \overleftrightarrow{DC}$. In $\triangle PP_2P_3$, therefore $\overleftrightarrow{DC}$ intersects $\overline{P_3P_2}$ in its midpoint, call it $\underline{X}$. Thus $DX = \frac{1}{2}PP_2 = DC$. Therefore $C = \underline{X}$, or C is the midpoint of $\overline{P_2P_3}$ and $P_3 \longrightarrow P_2$. If P is in $\overleftrightarrow{AB}$, essentially the same proof applies, for 2DC is still equal to PP_2 while $\overleftrightarrow{P_1P_2} \parallel \overleftrightarrow{AB}$ because the lines are the same.



5. Using a rectangular coordinate system with a as x-axis and P as origin, let A have coordinates (x, y),

$$A(x, y) \xrightarrow{R_a} A_1(x, -y) \xrightarrow{H_P} A_3(-x, y).$$

$$A(x, y) \xrightarrow{H_P} A_2(-x, -y) \xrightarrow{R_a} A_3(-x, y).$$

$$\therefore H_P R_a = R_a H_P.$$

6. The composition of four line reflections is a direct isometry. A glide reflection is an opposite isometry. A direct isometry cannot equal an opposite isometry.

7. Under all translations and all half-turns.

8. (a) identity transformation

   (b) line reflection

   (c) translations

   (d) rotations.

9. (a) The direct isometries are rotations (including half-turns) and translations. The opposite isometries are line reflections and glide reflections.

(b) No; a half-turn is an involution, but not an opposite isometry. It __may__ be an opposite isometry, for example, reflection is a line.

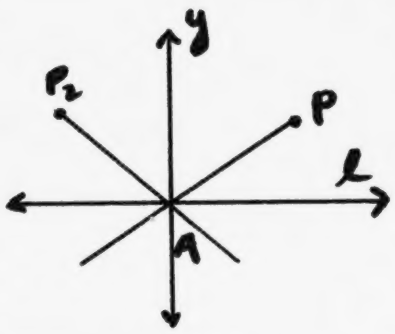
10. Let PQ be the x-axis and b the y-axis. Then (a, 0) may be assigned to Q, (-a, 0) to P and (x, y) to any point A.

$$A(x,\ y) \xrightarrow{R_b} A_1(-x,\ y) \xrightarrow{H_P} A_3(-2a+x,\ -y)$$

$$A(x,\ y) \xrightarrow{H_Q} A_2(2a-x,\ -y) \xrightarrow{R_b} A_3(-2a+x,\ -y)$$

$$\therefore\ H_P R_b = R_b H_Q.$$

11. Construct the y-axis through A



If P(x, y) then $P \xrightarrow{Ry} P_2$

$$(x,\ y) \xrightarrow{Ry} (-x, y)$$

Slope AP = $\frac{y}{x}$ and Slope AP$_2$ = $\frac{y}{-x} = -\frac{y}{x}$

Hence the reflection of AP in the y-axis has a slope which is opposite (add. inv.) to its slope.

Furthermore, since a half-turn does not alter the slope of a line

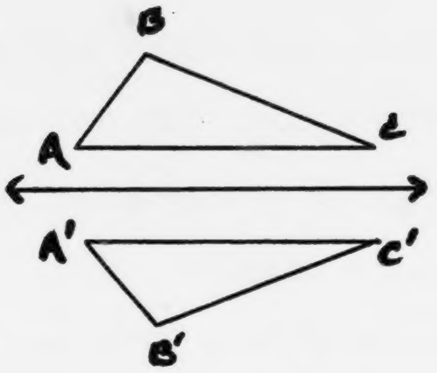
$R_m \circ H_B \circ R_\ell \circ H_A$ does not alter the slope of AP.

[slope AP$_2$ = - slope AP and slope BP$_4$ = -slope P$_2$B]

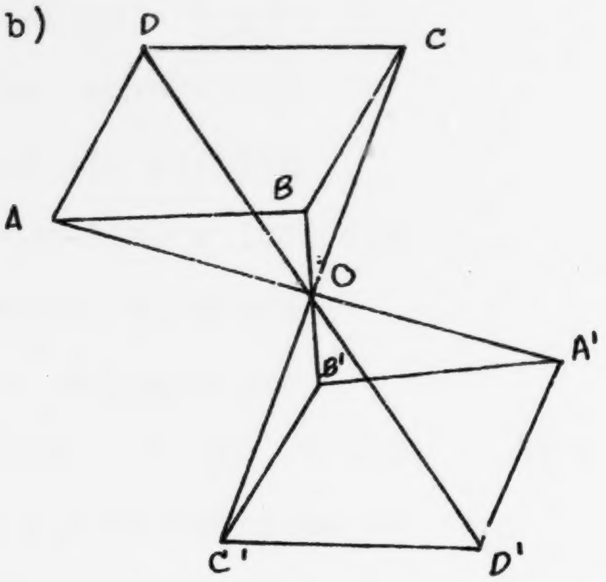
Hence AP || BP$_4$.

12. Two figures are congruent if an isometry maps one onto the other.

a)



b)



13. A figure has symmetry if it is invariant under an isometry.

    a. An isosceles triangle. (There are many possible answers).

    b. A rectangle. (Other answers are possible).

    c. A square. (Circle, any regular polygon).

Suggested Test Items

1. Make a drawing that shows the image of a given ΔABC under

    a. a line reflection in a line through C, not containing interior points of ΔABC.

    b. a half-turn in the midpoint of $\overline{AB}$.

    c. a rotation about A through ∠A, from $\overrightarrow{AB}$ to $\overrightarrow{AC}$.

    d. the translation that maps A onto the midpoint of $\overline{AC}$.

2. Describe an isometry that, in general has

    a. no fixed points

    b. has exactly one fixed point

c. has one fixed point and fixes all lines through that point.

d. fixes all points in one line and in lines perpendicular to that line.

3. Let the diagonal of parallelogram ABCD intersect in E. If F is in $\overline{AB}$ and $\overline{FE}$ intersects $\overline{CD}$ in G, show that CG = FA and $m\angle CGE = m\angle AFE$.

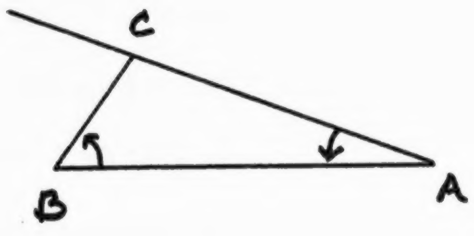
4. Show, that a glide reflection can be regarded as the composition of two half-turns and a line reflection.

5. Let ABCD be a rectangle. Under what isometries is the rectangle invariant? (there are four). Show that the set of these isometries with composition as operation is a group by displaying a group table.

6. Let lines a and b intersecting in O serve as axes for line reflections R_a and R_b.
Show R_aR_b in general does not equal R_bR_a.

7. The center of a circle is P. A line through P intersects the circle in points A and B. A second line through P intersects the circle in points C and D. Prove $\triangle APC \cong \triangle BPD$ and $m\angle PCA = m\angle PDB$.

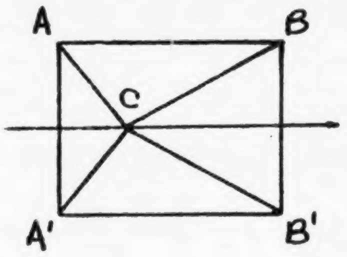
8. (An optional problem).



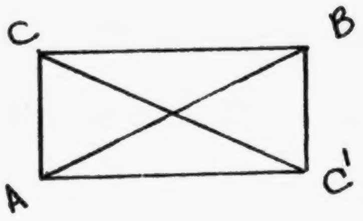
Let r_1 be a motion about point A and r_2 a rotation about point B. Assuming that r_1 can be decomposed into two line reflections whose axes are $\overline{AC}$ and $\overline{AB}$ in that order and r_2 can be decomposed into two line reflections whose axes are AB and BC, in that order, prove $r_2 \circ r_1$ is a rotation about C.

Answers for Suggested Test Items

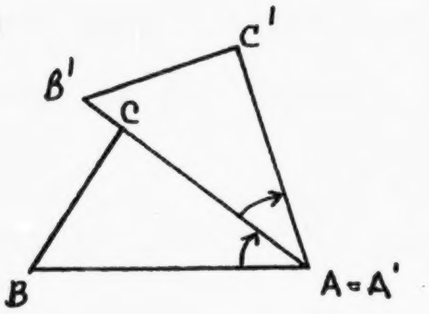
1. a)



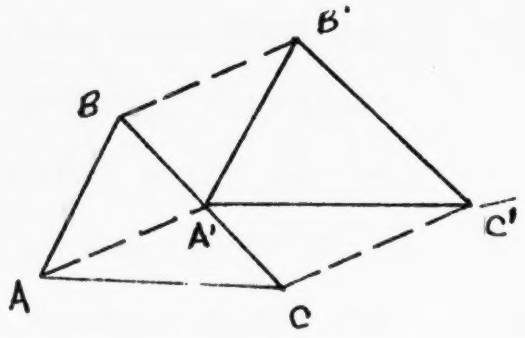
b)



c)



d)



2. a. A translation or a glide reflection
   b. A rotation
   c. A half-turn
   d. Line reflection

3. Under the half-turn about E A $\longrightarrow$ C, B $\longrightarrow$ D, F $\longrightarrow$ F'. Since (A,F,B) are collinear, so are (C, F',D). But both F' and G are in EF and DC. Therefore F' = G. Since a half-turn is an isometry, CG = FA and $m\angle CGE = m\angle AFE$.

4. A glide reflection can be regarded as the composition of a translation and a line reflection. But a translation can be regarded as the composition of two half-turns in distinct center. (See Exercise 11, Section 9,8).

5. The group table is

	i	R_a	R_b	H_E
i	i	R_a	R_b	H_E
R_a	R_a	i	H_E	R_b
R_b	R_b	H_E	i	R_a
H_E	H_E	R_b	R_a	i

6. This may be done by making a drawing that shows the image of a point under R_aR_b is not the same as the image of the same point under R_bR_a. But this does not merit full credit.

A general proof can be based on the Theorem that the composition of the two line reflections is a rotation from the first axis to the second. Reversing the order of axes reverses the orientation of the rotation and hence produces, in general, different images.

7. Using the half-turn in P, and knowing that the radii of a circle have the same lingth, $A \longrightarrow B$, $P \longrightarrow P$, $C \longrightarrow D$. Therefore $\triangle APC \longrightarrow \triangle BPD$ and $APC \cong \triangle BPD$. Since $\triangle PCA \longrightarrow \triangle PDB$, $m\angle PCA = m\angle PDB$ (isometries preserve angle measure).

8. $r_1 = R_{AB} \circ R_{AC}$, $r_2 = R_{BC} \circ R_{AB}$.

Therefore $r_2 \circ r_1 = R_{BC} \circ R_{AB} \circ R_{AB} \circ R_{AC}$

But $R_{AB} \circ R_{AB} = i$, and $R_{BC} \circ i = R_{BC}$,
Therefore $r_2 \circ r_1 = R_{BC} \circ R_{AC}$. Since BC and AC intersect in C, $r_2 \circ r_1$ is a rotation about C.

Chapter 10

Length, Area, Volume

Time Estimate for Chapter: 11 days

We consider measure to be a function. There is no difficulty in defining the domain of this function for the measurement of segments, nor is there any doubt that the range is the set of non-negative real numbers. This clarity is essentially due to the case of subdividing a given segment into segments, all of which are similar to it. However the situation is less clear in measuring regions. This is partly due to the fact that a region cannot always be subdivided into regions that are congruent to a unit region, nor even similar to it. For this reason we prefer to discuss first the simple case of the areas of rectangular regions, and only after some basic notions have been discussed do we consider the area of other regions in Section 10.13, and of circular regions in Section 10.16. The case for volumes of solids is still less clear and we restrict ourselves to the volume of rectangular solids (Sections 10.7).

In constructing a measure function we start with a domain of the set of figures we are to measure. This can be a set of easily measurable figures, such as rectangles or polygons or rectangular solids. These domains can be extended later. Then we postulate that we can assign a non-negative real number to each figure such that the same number is assigned to any two congruent figures (sometimes called the property of invariance), and to every figure that is subdivided there is assigned a number that is the sum of the numbers assigned to the subdivisions (the additive property), and finally, the number assigned to any figure is determined when a measure is assigned to one figure in the set. It should be noted that different measures may have different sets of real numbers as their ranges or co-domain. For example, the range of angle measure is $\{\theta : \theta \in R \text{ and } 0 \leq \theta < 180\}$; the range for probability measure is $\{p : p \in R \text{ and } 0 \leq p \leq 1\}$; the range for measures of segments, areas, and volumes is $\{m : m \in R \text{ and } m \geq 0\}$. We have also mentioned the possibility that a measure does not exist, but it is not important at this level of study.

In Section 10.9 and 10.11 we extend the domain of the area function from the set of rectangular regions to the set of triangular and quadrilateral regions. We hint, but do not develop, the possibility of extending it still further to include polygonal regions, that is, regions that can be subdivided into triangular regions. The interesting feature of this extension is that it is accomplished by deductions. The theorems in these sections give formulas for finding areas. You should not expect proofs to be rigorous.

Overhead projection can be used effectively to show the approximation process in operation. For instance, in Section 10.4, where segments are being measured, a set of overlays, each with smaller subdivisions, would show nicely a set of lower and upper approximations. The same is true in Section 10.5 where rectangular regions are measured, and in Section 10.13 where the area of a map is measured.

References: Kutuzov, B.V. Studies in Mathematics.
Geometry School Mathematics Study Group, Vol. IV, 1960.

Levi, Howard. Foundations of Geometry and Trigonometry. Englewood Cliffs: Prentice-Hall, 1960.

The number of teaching days for this unit may vary depending on the background of the student. The total teaching time alloted should not exceed 12 days.

10.1, 10.2 Introduction and measure on Sets
(Time estimate including 10.3 = 1 day)

These two sections serve as a general introduction to the measurement of line segments, planar regions and solids. Section 10.2 develops the general idea of the counting measure of a finite set and the property of additivity of measures. These are general principles and will be used again in a more complete development of the counting principle.

10.3 Answers

$n(A) = 16$

$n(B) = 9$

$n(C) = 8$

$n(D) = 4$

2. $n(A \cap B) = n(\{x : x \in X \text{ and } x \text{ is a multiple of } 3 \text{ and } x \text{ is a multiple of } 5\})$

   $= n(\{x : x \in x \text{ and } x \text{ is a multiple of } 15\}) = 3.$

   $n(A \cap C) = n(c) = 8.$

   $n(A \cap D) = 1.$

   $n(B \cap C) = 1.$

   $n(B \cap D) = n(\emptyset) = 0.$

   $n(C \cap D) = n(\emptyset) = 0.$

3. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 16 + 9 - 3 = 22.$

   $n(A \cup C) = n(A) = 16.$

   $n(A \cup D) = 19.$

   $n(B \cup C) = 16.$

   $n(B \cup D) = 13.$

   $n(C \cup D) = 12.$

4. $n(A \times B) = n(A) \cdot n(B) = 16 \cdot 9 = 144.$ $n(B \times C) = 72.$

   $n(A \times C) = 128.$ $n(B \times D) = 36.$

   $n(A \times D) = 64.$ $n(C \times D) = 32.$

5. $n(A \times B \times D) = 16 \cdot 9 \cdot 4 = 576.$

6. $n(A \cap B \cap d) = n(\{x : x \in X \text{ and } x \text{ is a multiple of } 3 \text{ and } x \text{ is a multiple of } 5 \text{ and } x \text{ is a multiple of } 11\})$

   $= n(\emptyset) = 0.$

   $$\begin{aligned} n(A \cup B \cup D) &= n(A \cup B) + n(D) = n[(A \cup B) \cap D] \\ &= 22 + 4 - n[(A \cap D) \cup (B \cap D)] \\ &= 22 + 4 - [n(A \cap D) + n(B \cap D) - n(A \cap B \cap D)] \\ &= 22 + 4 - [1 + 0 - 0] = 26 - 1 = 25. \end{aligned}$$

10.4 Lengths of Line Segments (Time estimate = 1/2 - 1 day)

It will be recalled that Chapter 5 contained a rather complete discussion of the measuring process of line segments. Therefore, it is only necessary to review quickly this procedure and the various principles of this section.

10.5 Areas of Rectangular Regions (Time estimate including 10.6 = 1 day)

This section develops the general technique for finding the measure of a rectangular region. Although students may be familiar with the formulas $A = \ell w$ and $P = 2\ell + 2w$, they should understand the process leading to them.

10.6 Answers

1. a. 21 sq. in. b. 21 sq. in. c. 21 sq. in.
   d. 13.12 sq. in. e. $\sqrt{6}$ sq. in. f. 30 sq. in.
   g. 8 sq. in. h. 3.3 sq. in. i. 2 sq. in.
2. A square is a rectangle. Therefore $K_{square} = s.s = s^2$

   or $K = \ell w$, $K = s \cdot s$, $K = s^2$
3. a. $K_{ADEH} = 15 \cdot 6$ or 90 b. $K_{ACMN} = 10 \cdot 2 = 20$
   c. $K_{ABGH} = 5 \cdot 6 = 30$ d. $K_{BCLK} = 5 \cdot 4 = 20$
   e. $K = 10 \cdot 4 = 40$

4. a. Two rectangular regions have the same area if they are congruent.

   b. The area of a rectangular region is equal to the sum of the areas of its subdivisions.

   c. $K_{BCLK} = 20$, $K_{ADEH} = 90$, and $20 = \frac{2}{9} \cdot 90$.

   d. $K_{ADEH} = K_{ACFH} + K_{CDEF}$ (The additive principle)

   $\therefore K_{ADEH} - K_{CDEF} + K_{ACFH}$ (addition property of =)

5. a. $K_{R_1} = a^2$ b. $K_{R_2} = ab$ c. $K_{R_3} = ab$ d. $K_{R_4} = b^2$

   e. The area of the entire square region is $(a+h)^2$. But by algebraic principles $(a+h)^2 = a^2 + 2ah + h^2$, and this is $K_{R_1} + K_{R_2} + K_{R_3} + K_{R_4}$.

   $(a+h)^2 = a^2 + ah + ah + h^2 = a^2 + 2ah + h^2$

6. a. An adjacent side is $\sqrt{5^2-4^2}$ inches long or 3 inches. $K = 4 \cdot 3 = 12$ sq. in.

   b. An adjacent side is $\sqrt{41-25}$ or 4 in. long. $K = 4 \cdot 5 = 20$ sq. in.

   c. $\sqrt{13^2 - 12^2} = 5$. $\therefore K = 5 \cdot 12 = 60$ sq. in.

   d. $\sqrt{10^2 - 6^2} = 8$ $\therefore K = 8.6 = 48$ sq. in.

   e. $\sqrt{11-5} = \sqrt{6}$ $\therefore K = \sqrt{5} \cdot \sqrt{6} = \sqrt{30}$ sq. in.

   f. $\sqrt{25^2 - 15^2} = 20$ $\therefore K = 20 \cdot 15 = 300$ sq. in.

7. a. Since ABCD is a rectangle, $\Delta ACD \cong \Delta CAB$ ($\Delta ACD \rightarrow \Delta CAB$ by a half turn about the midpoint of AC)

   $K_{ACD} = K_{CAB}$, by the congruence principle for areas.

   b. By the additive principle $K_{ABCD} = K_{ACD} + K_{CAB} = 2K_{ACD}$

   $\therefore K_{ACD} = \frac{1}{2} K_{ABCD}$.

8. Under the half-turn about the midpoint of $\overline{PR}$ $\triangle PRS \rightarrow \triangle RPQ$. Hence $\triangle$ PRS $\cong$ $\triangle$RPC and, by the congruence principle, $K_{PRS} = K_{RPQ}$. By the additive principle $K_{PQRS} = K_{PRS} + K_{RPQ} = 2K_{PRS}$.

9. $K_{PTQ} = K_{RTS}$.

10. The statement is false as indicated by the following counter example: The rectangular region with dimensions 6 by 2 has the same area as the rectangular region with dimensions 4 by 3, but they are congruent.

11. (Drawing figures at the board would be helpful to the students)
    a. Each side of a square foot is 12 inches long. Its area is therefore 12·12 square inches.
    b. Each side of a square yard is 3 feet long. Its area is 3·3 sq. ft.
    c. Each side of a square centimeter is 10 millimeters long. Its area is 10·10 square millimeters.

12. The area of the larger is 4 times the area of the smaller.

$$K_{ABCD} = \ell w \qquad \begin{aligned} K_{PQRS} &= (2\ell)(2w) \\ &= (2\ 2)(\ell w) \quad \text{Associativity} \\ &= 4\ell w \end{aligned}$$

$\therefore$ K_{PQRS} is 4 times the area of K_{ABCD}

13. a. $K_1 : K_2 = \ell_1 \cdot w_1 : \ell_1 \cdot 2w_1 = 1:2$

    b. $K_1 : K_2 = \ell_1 \cdot w_1 : 2\ell_1 \cdot 3w_1 = 1:6$.

    c. $K_1 : K_2 = \ell_1 \cdot w_1 : \frac{1}{2} \ell_1 \cdot 2w_1 = 1:1$

    d. $K_1 : K_2 = \ell_1 w_1 : \frac{1}{3}\ell_1 : 4w_1 = 3:4$

14. a. Let s = length of side of square. By the Pythagorean property of right triangles $s^2 + s^2 = 12^2$ or $2s^2 = 144$

$\therefore s^2 = K = 72$ sq. in.

b. $s^2 + s^2 = (8\sqrt{2})^2$, $2s^2 = 64.2$ $s^2 = K = 64$ sq. ft.

c. $s^2 + s^2 = (6\sqrt{2})^2$, $2s^2 = 36.2$ $s^2 = K = 36$ sq. ft.

15. Let s = length of a side. Then $s^2+s^2 = d^2$ or $s^2 = K = \frac{1}{2}d^2$.

10.7 Volumes of Rectangular Solids (Time estimate = independent or 1 day)

Since the method for finding the volume of a rectangular solid is analogous to that used in Section 10.5, students should experience little difficulty with this section. The teacher may want to assign this section for homework to leave extra time in class for going over the exercises.

10.8 Exercises (Time estimate = 1 day for exercises 10.6 and 10.8)

1. a. $V = 3\cdot 4\cdot 2 = 24$ cu. ft. b. $V = \frac{5}{2}\cdot 4.5 = 50$ cu. ft.

c. $V = \sqrt{2}\cdot\sqrt{3}\cdot 2 = 2\sqrt{6}$ cu. ft. d. $\ell = 3.1 \cdot 2.3 \cdot 4 = 28.52$ cu. ft.

2. The volume of the first box is $2 \cdot \frac{3}{2} \cdot 1 = 3$ cu. ft.

The volume of the second box is $1\frac{3}{4} \cdot 1\frac{3}{4} \cdot 1 = 3\frac{1}{16}$ cu. ft.

$\therefore$ The second box is larger by $\frac{1}{16}$ cu. ft.

3. Volume.

4. a. length

b. area

c. volume

5. Each edge of the cubic is e. $\therefore V = e\cdot e\cdot e = e^3$.

6. a. Each edge of a cubic foot is 12 inches long. Therefore its volume is $12^3 = 1728$ cu. in.

   b. Each edge of a cubic yard is 3 feet long. The volume of a cubic yard is 3^3

   c. The number of cubic decimeters in a cubic meter is $10^3 = 1000$.

   d. There are 1760 yards in one mile. Therefore there are 1760^3 cubic yards in a cubic mile. To approximate 1760^3 we can take $1700 \cdot 1700 \cdot 1800$ or $289 \cdot 18 \cdot 10^6$ or $300 \cdot 18 \cdot 10^6$ which is $54 \cdot 10^8$ $\therefore$ 1760^3 is about $54 \cdot 10^8$.

7. (a) $V = 3 \cdot 2 \cdot 5 = 30$ (b) $V = 4 \cdot 3 \cdot 5 = 60$

10.9 Area of Triangular Regions (Time estimate including 10.10 = 1 day)

Students should be challenged to find the formula for the area of a triangular region. After the informal discussion, the derivation of Formula 2 can be done in more detail.

10.10 Exercises (Drawings will be helpful in many of these exercises)

1. a. $K = \frac{1}{2} \cdot 8 \cdot 12 = 48$ sq. in. b. $K = \frac{1}{2} \cdot 8 \cdot 12 = 48$ sq. in.

   c. $K = \frac{1}{2} \cdot 8 \cdot 12 = 48$ sq. in. d. $K = \frac{1}{2} \cdot 3 \cdot 4 = 6$ sq. ft.

   e. $K = \frac{1}{2} \cdot 12 \cdot 25 = 150$ sq. in.

f. $5^2 + cb^2 = 13^2$

$25 + cb^2 = 169$

$cb^2 = 144$

$cb = 12$ cm.

Then $K = \frac{1}{2} \cdot 5 \cdot 12. = 30$ sq. cm.

2. The diagram in the book is inaccurate (notice origin of altitude $\overline{AD}$); therefore, it would be wise to have students ignore the direction to measure the figure in text and have them draw one of their own. The exercise should lead to the conclusion that the area of the triangle is the same regardless of which altitude and base is selected.

3. By the Pythagorean property of Right Triangles we have:

$a^2 + b^2 = c^2$

$10^2 + b^2 = 26^2$

$100 + b^2 = 676$

$b^2 = 576$

$b = 24$

Then $K = \frac{1}{2} \cdot 24 \cdot 10 = 120$ sq. in.

4. By the Pythagorean property of Right Triangles we have:

$a^2 + b^2 = c^2$

$a^2 + \ell^2 = h^2$ SPE

$a^2 = h^2 - \ell^2$ Field Theorem

$a^2 = \sqrt{h^2 - \ell^2}$ Length of a

Then $K = \frac{1}{2}ab$

$K = \frac{1}{2}\ell(\sqrt{h^2 - \ell^2})$ SPE (where ℓ is the altitude and $\sqrt{h^2 - \ell^2}$ is the base.

5. By the Pythagorean property

$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

Let x = length of the altitude to the hypotenuse, then

$K = \frac{1}{2} \cdot x \cdot 10$ and $K = \frac{1}{2} \cdot 6 \cdot 8$

Then $\frac{1}{2} \cdot x \cdot 10 = \frac{1}{2} \cdot 6 \cdot 8$ See exercise 2 above, and by transitive property.

$$5x = 24$$
$$x = 4.8 \text{ or } 4\ 4/5$$

6. $K_{ABC} = \frac{1}{2} \cdot 20 \cdot 40 = 400$: $K_{ADC} = \frac{1}{2} \cdot 30 \cdot 40 = 600$. $K_{ABCD} = 400 + 600 = 1{,}000$.

7. Let x be the length of the altitude to the base

$10^2 = 4^2 + x^2$, $x = \sqrt{84}$, $K = \frac{1}{2} \cdot \sqrt{84} \cdot 8 = 4\sqrt{84}$ or $8\sqrt{21}$

8. a. Let ABC be the triangle and $\overline{AD}$ an altitude.

$(AD)^2 + 6^2 = 12^2$, $AD = \sqrt{108}$ or $6\sqrt{3}$

(regardless of which $\overline{AD}$ is selected.)

b. $K = \frac{1}{2} \cdot 12 \cdot 6\sqrt{3} = 36\sqrt{3}$

9. Let ABC be the triangle and $\overline{AD}$ an altitude. Then

$$AD^2 + \frac{s}{2}^2 = s^2$$
$$AD^2 + \frac{s^2}{4} = s^2$$
$$AD^2 = s^2 - \frac{s^2}{4}$$
$$AD^2 = \frac{3s^2}{4}$$
$$AD = \frac{s\sqrt{3}}{2}$$

Then $K_{ABC} = \frac{1}{2}ab$

$K_{ABC} = \frac{1}{2}\,\frac{s\sqrt{3}}{2}\,s$

$K_{ABC} = \frac{s^2\sqrt{3}}{4}$ or $\frac{s^2}{4}\sqrt{3}$

10.11 Areas of Parallelograms and Trapezoidal Regions

(Time estimate including 10.12 = 2 days)

As in section 10.9, students will enjoy trying to find (on their own) formulas for the areas of parallelograms and trapezoidal regions. The teacher should ellicit, from the class, the various steps leading to the derivation of the formulas.

10.12 Answers to Exercises

1. a. $K = 7 \cdot 10 = 70$ b. $K = 9 \cdot 7 = 63$ c. $K = 4 \cdot 10 = 40$

2. a. $K = \frac{1}{2} \cdot 5\,(8+10) = 45$ b. $K = \frac{1}{2} \cdot 6\,(6+12) = 54$

   C . $K = \frac{1}{2} \cdot 7(10 + 20) = 105$

3. a. Trapezoid. $K = \frac{1}{2} \cdot 5(8+5) = \frac{65}{2}$

   b. Parallelogram. $K = 8 \cdot 6 = 48$

   c. Square. $S = 3\sqrt{2} \cdot K = (3\sqrt{2})^2 = 18$

   d. Trapezoid. $K = \frac{1}{2} \cdot 5\,(8+6) = 35$

   e. Trapezoid. $K = \frac{1}{2}5\,(8+4) = 30$

   f. Trapezoid. $K = \frac{1}{2}8\,(4+7) = 44$

4. a. $K = 8\cdot 10 = 80$. Cost $=(80)(.15)= \$12.00$
   b. $K = \frac{1}{2}\cdot 10(10+14) = 120$. Cost $= 120\cdot .15 = \$18.00$
   c. $K = \frac{1}{2}\cdot 6\cdot 5 = 15$ Cost $= 15\cdot 15 = \$2.25$
   d. $b = 16$, $K = 15\cdot 16 = 240$. Cost $240\cdot .15 = \$36.00$
   e. $a = \sqrt{20^2-6^2} = \sqrt{364}$. Cost $= .15\sqrt{364}$ (about \$2.86)
   f. $K = \frac{1}{2}(10\sqrt{2})^2 = 100$ Cost $= .15 \times 100 = \$15.00$

5. a. 4:1 b. 9:1 c. 4:9 d. $k^2:1$
6. a. 4:1 b. 9:1 c. 4:25 d. $k^2:1$

10.13 Areas of other Regions (Independent assignment)

Since this section is interesting and not difficult, it should be assigned independently. If students have questions they can be answered when going over homework exercises.

10.14 Circumference of a Circle and π (Time estimate including 10.15 = 1 day)

Before doing this section, the teacher may want to use a discovery approach as outlined below:

Prior to beginning the lesson, the teacher should have each student bring a tin can and some string to class. The teacher can begin the lesson by introducing or reviewing the definitions of circle, radius, diameter and circumference. Using one end of the tin can as a model of a circle, the students should find the circumference by pulling the string tightly around the can and then measuring the length of the string. After measuring the diameter of the circle as best he can, the student should compute the ratio $\frac{c}{d}$. On the board, the teacher should have prepared a chart with columns entitled c, d and $\frac{c}{d}$. After having several pupils list their results on the board it will become clear that $\frac{c}{d}$ is about the same in each case. At this point, the teacher may wish to include some historical reference by mentioning that the ancient Greeks also noticed this constant relationship between the circumference and diameter of a circle.

The teacher can now return to section 10.14 and complete the lesson by explaining the procedure outlined in the text. (It should be pointed out that paragraph 2 on p. 219 will have more meaning after this intuitive approach.)

10.15 Answers to Exercises

1. a. 20π b. 16π c. 2π d. π e. $2\sqrt{3}\,\pi$
2. a. 38 in. b. 75 yd. c. 628 m d. 11 cm
3. a. 44 in. b. 22 ft. c. 8800 in. d. 176 m
4. 2:1
5. a. 12π in. b. 50π ft. c. $\frac{1}{2}\pi$ yd. d. $.1\pi$ ft.
6. a. 10π in. b. 25π ft. c. $\frac{1}{2}d\,\pi$.
7. Side of square = $10\sqrt{2}$, perimeter = $40\sqrt{2}$, about 56.56.
   $C = 2\cdot\pi\cdot 10 = 20\pi$, about 62.84
   62.84 - 56.56 = 6.28 or 6 in.
8. Let d = diameter's length. $d^2 = 3^2+5^2$, $d = \sqrt{34}$ $C = \pi\sqrt{34}$.

9. a. 12 b. $\frac{33}{2}$ c. $\frac{12}{\pi}$ d. k

10. a. 18 b. 4 c. $\frac{18}{\pi}$ d. 2k

10.16 Areas of Circular Regions
(Time estimate including 10.17 = 1 day)

As in section 10.14, the teacher may want to give a more intuitive approach to the area of a circle before considering the approximation technique of this section. This can be done by having the students cut a circular region into 6 congruent regions as in figure 1. The students should then rearrange the regions as in figure 2.

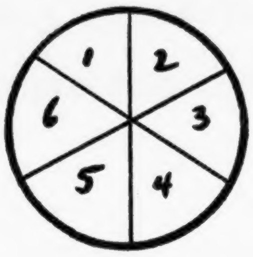


Figure 1

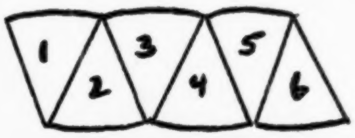


Figure 2

It will be noticed that figure 2 is a very rough approximation of a parallelogram. Indeed, if we cut the circle into a very large number of congruent regions, we would have a closer approximation to a parallelogram. Thus, the area of the circle is approaching the area of a parallelogram.

If the circumference = c, and the radius = r, the base of the parallelogram = $\frac{c}{2}$, and the height = r. Therefore

$$\begin{aligned} K &= b \cdot h \\ &= \frac{2\pi r}{2} \cdot r = \pi r^2 \end{aligned}$$

10.17 Answers to Exercises

1. a. 100π b. 64π c. π d. $\frac{1}{4}\pi$ e. 3π

2. a. 113.04 b. 200.96 c. 314 d. 15.70

3. a. 154 sq.in. b. $\left\{\begin{array}{l}\text{38 1/2 sq. ft. or}\\ \text{38.5 sq. ft.}\end{array}\right.$ c. 61600 sq.yd. d. 22 sq.yd.

4. a. π b. $\frac{1}{4}\pi$ c. $\frac{25}{4}\pi$ d. 3π

5. a. 5 b. 8 c. $\frac{7}{\sqrt{\pi}}$ d. $\sqrt{\frac{20}{\pi}}$ or $2\sqrt{\frac{5}{11}}$

6. a. $\pi r^2 = 25\pi$ implies $r = 5$. $\therefore C = 10\pi$
   b. $r = 2$, $C = 4\pi$
   c. $r = 1$, $C = 2\pi$
   d. $r = \frac{1}{2}$, $C = \pi$

7. a. $2\pi r = 16\pi$ implies $r = 8$. $K = 64\pi$
   b. $r = 13$, $K = 169\pi$
   c. $r = 4$, $K = 16\pi$
   d. $r = \frac{4}{\pi}$, $K = \pi\left(\frac{4}{\pi}\right)^2 = \frac{16}{\pi}$

8. a. $\frac{1}{2}(\pi\, 5^2) = \frac{25}{2}\pi$ or 12.5π b. $\frac{1}{2}(\pi 4^2) = 8\pi$
   c. $\frac{1}{2}(\pi 50^2) = 1250\pi$ d. $\frac{1}{2}(\pi 3^2) = \frac{9}{2}\pi$

9. a. Let the respective radii be $2r$ and r.
   $C_1 : C_2 = 4\pi r : 2\pi r = 2:1$
   b. $K_1 : K_2 = 4\pi r^2 : \pi r^2 = 4:1$

10. a. 3:1, 9:1 b. 3:2, 9:4
    c. 4:3, 16:9 d. 5:1, 25:1

    The teacher should generalize at this point that
    $c_1 : c_2 = r_1 : r_2$ and $K_{c_1} : K_{c_2} = (r_1)^2 : (r_2)^2$

11. a. $AD = 10\sqrt{2}$, $K_{ABCD} = (10\sqrt{2})^2 = 200$.

    b. $K_{circle} = 100\pi$

    c. This region has an area which is one-fourth of the difference between 100π and 200. It is $25\pi - 50$.

    d. This region has an area which is three-fourths of the difference between 100π and 200. It is $75\pi - 150$.

10.19 Review Exercises

1. a. The sum of the lengths of the subdivisions of a segment is equal to the length of the segment.

   b. The sum of the areas of the subdivisions of a region (if they have areas) is the area of the region.

   c. The sum of the volumes of the subdivisions of a solid (if they have volumes is the volume of the solid.

2. Let s = length of a side $s^2 + s^2 = 8^2$, $s^2 = 32$

   $s = \sqrt{32}$. $K = \frac{1}{2}\sqrt{32}^2 = 16$.

   Alternate solution. The area of the triangular region is one half the area of the square region whose diagonal measures 8 in. $K_{square} = \frac{1}{2}8^2 = 32$. $K_{\Delta} = \frac{1}{2}\cdot 32 = 16$.

3. a. $C = 16\pi; K = 64\pi$ b. $C = 10\pi$, $K = 25\pi$

   c. $C = 24\pi$, $K = 144\pi$

4. $\pi r^2 = 100\pi$. $r = 10$. $C = 20\pi$

5. The length of one side is $6\sqrt{2}$. Area = 72

6. Let $\overline{AD}$ be the median. Triangles ADB and ADC have same length bases in $\overline{DB}$ and $\overline{DC}$; and a common altitude from A to BC.

7. Using the result of Exercise 9.

   a. $K_{ADB}: K_{ADC} = 1:1$

8. a. Trapezoid $K = \frac{1}{2} 2(7+2) = \frac{27}{2}$

   b. Parallelogram $K = 6 \cdot 5 = 30$.

   c. Two triangles with common base in x-axis $K = \frac{1}{2} 5 \cdot 3 = \frac{15}{2}$

   d. Two triangles, with common base whose endpoints are $(-3, 0)$ and $(4, 0)$. $K = \frac{1}{2} 4 \cdot 7 + \frac{1}{2} 5 \cdot 7 = \frac{63}{2}$

9. $V = 3 \cdot 6 \cdot 2 = 36$ cu. in.

10. $V = 3 \cdot 5 \cdot 6 = 90$.

Suggested Chapter Test Items

1. In a rectangular coordinate system consider the following set of points with their coordinates:

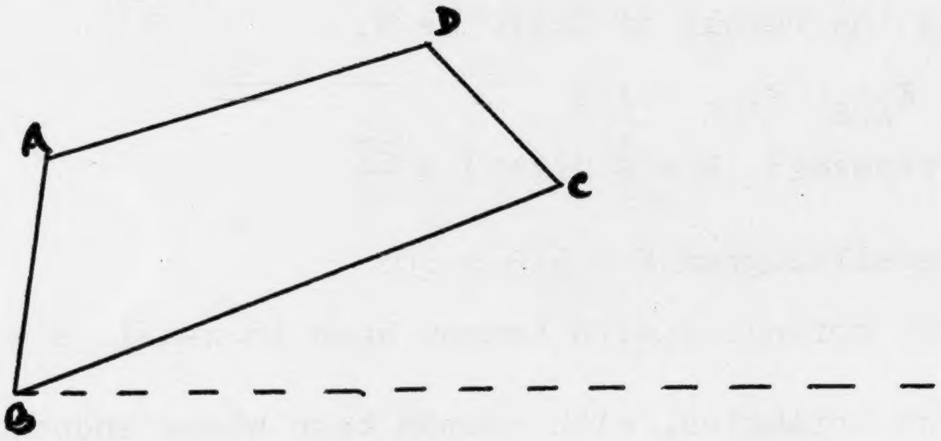
   $A(-2,0)$, $B(4,0)$, $C(6,3)$, $D(3,3)$, $E(0,3)$

   Find a. K_{AEB} b. K_{ABCE} c. K_{ABDE} d. K_{BCE}

2. The perimeter of a square is 24. Find the area of its region.

3. In surveying field ABCD, an east-west line was laid out through B, as shown. Then perpendicular to this line, lines from A,D,C intersected it in A^1, D^1, C^1. Find K_{ABCD} if $AA^1 = 12$ yds., $DD^1 = 20$ yds., $A^1B = 6$ yds., $BD^1 = 18$ yds., $D^1C^1 = 8$ yds., $CC^1 = 9$ yds.

(See figure next page.)



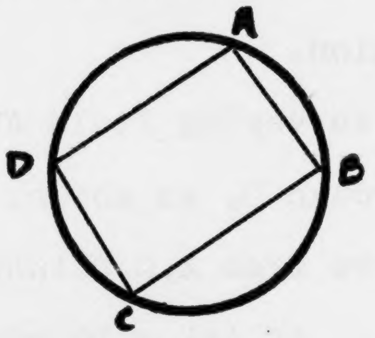
4. Let ABCD be a trapezoid
   $AB \parallel CD$, and $\overline{AC} \cap \overline{BD} = \{O\}$.
   Prove: a. $K_{ADC} = K_{BDC}$
   b. $K_{AOD} = K_{BOC}$

5. Find the radius of a circle whose circumference measure is equal to the measure of its region.

6. The area of a semicircular region is 18π. Find the radius of the related circle.

7. The diameter of a wheel is 21 inches. How many revolutions does it make covering 660 feet? (Use $\frac{22}{7}$ for π).

8. In the figure points A,B,C,D are on the circle whose radius is 10. ABCD is a square. (Assume BD is a diameter). Find:
   a. K_{circle}
   b. K_{square}
   c. The area of the region bounded by $\overline{DC}$ and the part of the circle that is on the opposite side of DC as A.



Answers for Chapter Test Items

1. a. $K_{AEB} = \frac{1}{2}\,3\cdot 6 = 9$ sq. units.

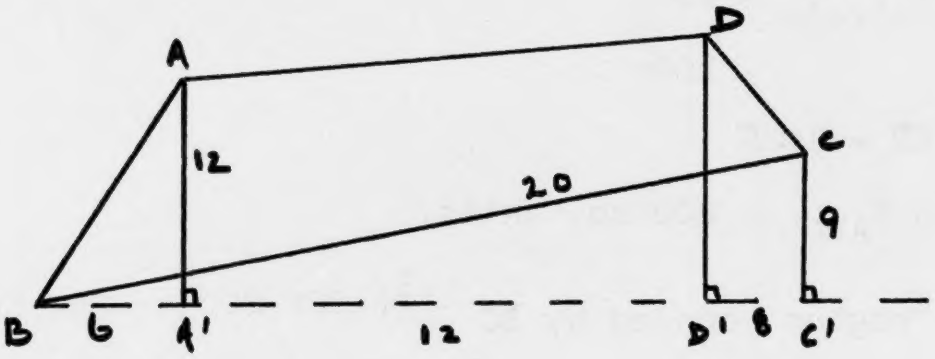
   b. $K_{ABCE} = 6\cdot 3 = 18$ sq. units.

   c. $K_{ABDE} = \frac{1}{2}\cdot 3\ (6+3) = \frac{27}{2}$ or $13\frac{1}{2}$ sq. units.

   d. $K_{BCE} = \frac{1}{2}\cdot 3\cdot 6 = 9$ sq. units.

2. Each side has measure 6. $K = 6^2 = 36$ sq. units.

3.



$$K_{ABCD} = K_{AA'DD'} + K_{DD'C'C} + K_{ABA'} - K_{BCC'}$$
$$= \frac{1}{2}(12)(12+20) + \frac{1}{2}(8)(20+9) + \frac{1}{2}(12)(6) - \frac{1}{2}(9)(26)$$
$$= 192 + 116 + 36 - 117$$
$$= 227 \text{ sq. yds.}$$

4. a. The altitude from A to $\overline{DC}$ = the altitude from B to $\overline{DC}$ and DC = DC $\therefore K_{ADC} = K_{BDC}$.

   b. $K_{ADC} = K_{AOD} + K_{DOC}$, $K_{BDC} = K_{BOC} + K_{DOC}$

   by (a) $K_{AOD} + K_{DOC} = K_{BOC} + K_{DOC}$

   $\therefore K_{AOD} = K_{BOC}$

5. Let r be radius. $2\pi r = \pi r^2$.

Since $r \neq 0$, $r = 2$.

6. $K_{circle} = 36\pi$, $\pi r^2 = 36\pi$, $r = 6$.

7. The circumference is $C = \pi d$

$$C = \frac{22}{7} \cdot 21$$

$C = 66$ This is the distance covered in one revolution.

The required number of revolution is $\frac{660}{66} = 10$ revolutions.

8. a) $K_{circle} = \pi r^2$

$= 100\pi$

b) $\overline{AB} = 10\sqrt{2}$

$\therefore K_{ABCD} = 200$ sq. units.

c) $K_{\text{region bounded by } \overline{DC}} = \frac{1}{4}(100\pi - 200) = 25\pi - 50$ sq. units.

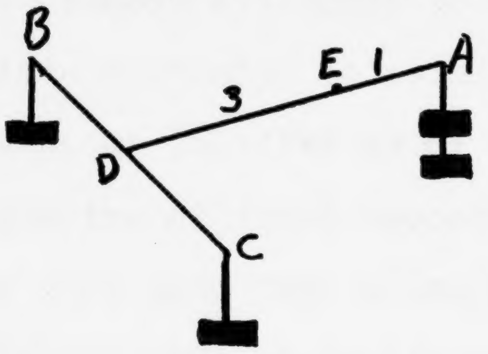
$K_{\text{region side of } DC} = \frac{3}{4}(100\pi - 200) = 75\pi - 150$ sq. units.

APPENDIX A
Mass Points

In preceding chapters we have used induction and deduction as the pedagogic situation seemed to dictate. Where deduction was simple we preferred it to induction or experimentation. It was one of our purposes in those chapters to give students many experiences with deductions. In this chapter we want to take our first step in the direction of formal deductions. But that step should not be a long one lest we overwhelm students with the many difficulties inherent in the process. The important objective is to make students aware of deductive proofs; that they appreciate the place of postulates of primitive statement in such proofs; that they learn to be critical in judging validity of proofs; that they become adept in writing simple proofs about mass points; and finally that they will come to enjoy the pleasures of the intellect in this activity.

The topic of mass points is related to centers of gravity and centroids. This might be made clear to students by performing experiments. The first of these experiments could easily be in connection with the definition of addition of mass points, as indicated in the see-saw diagram.

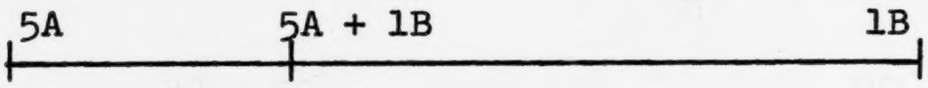
In Section 13.2 when Theorem 1 has been learned a second experiment can be performed to show that a cardboard triangle can be balanced on a pin at the point where the medians meet, that is, at the centroid. If you perform this experiment be sure to use a stiff cardboard that is homogeneous (as mortar). In setting the triangle to rest on the point of the pin, the pin should pierce the cardboard ever so slightly, to prevent slipping. If the experiment were performed under ideal conditions there should be no slipping. But no cardboard is ideally homogeneous, nor is the location of the centroid perfect. So cheat a little but be frank with your students. In this experiment you are considering the case of 1A + 1B+ 1C for which the same weight is set at each vertex. This is equivalent to using a homogeneous cardboard. The case of 2A + 1B + 1C is more difficult. However, it can be approximated. Connect A, B, and C with the two light (balsa) strips of wood, or two metal strips (from a coat hanger?), as shown at the right. At A,B,C suspend equal weights and support the system at E, such that DE:EA = 3:1. (Shades of mobiles). The weights at A should be a little more than twice the weight of the frame on one side of D. If this experiment is carried out, even though it is not as successful as you might like, it will help clarify the meaning of mass points and their importance in the engineering problems of equilibrium.



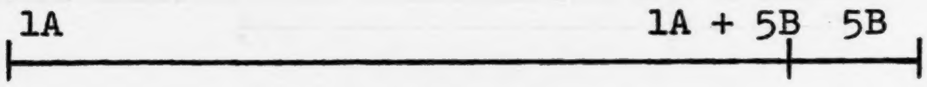
You will have to face the question of what degree of formality you will require in your students' proofs. Strictly speaking, the question is not one of degree, since a proof is either formal or it is not. The question is one of format or style. In one style, commonly called the two column form, a reason is expected for each statement in the proof. This is the goal which you should set for your students and when they take their chapter test they should be able to do this. (See Problem 3 in the Suggested Items for Chapter Test). However at the outset, when a student is giving a proof orally he should not be interrupted, nor corrected, until he has completed his effort. It is likely that students will omit some reasons, particularly the obvious ones. They have this in common with mathematicians and should not be too severly criticized for this kind of omission. However, in order that the nature of deductive proof be clear, it will eventually be necessary to require all reasons. This requirement, however, should be imposed slowly, and only when students are ready for it.

A.3 Answers

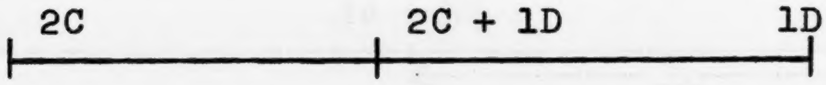
1. a.



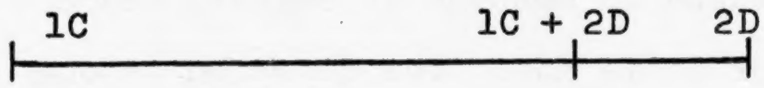
b.



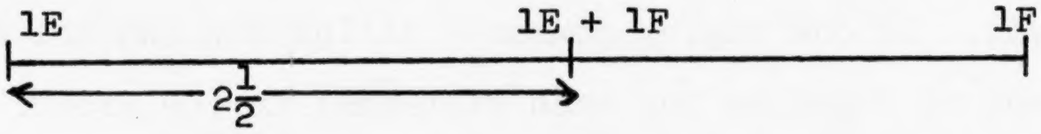
c.



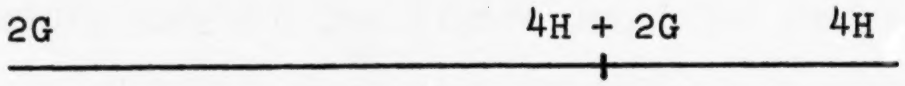
d.



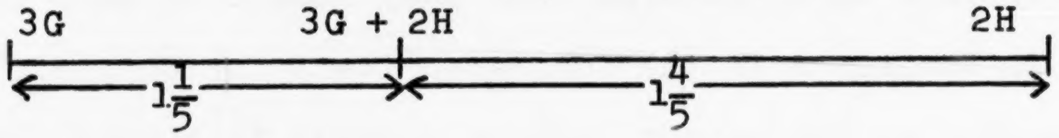
e.


1E
1E + 1F
1F
$2\frac{1}{2}$

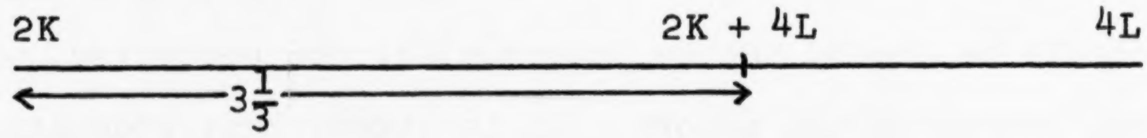
f.



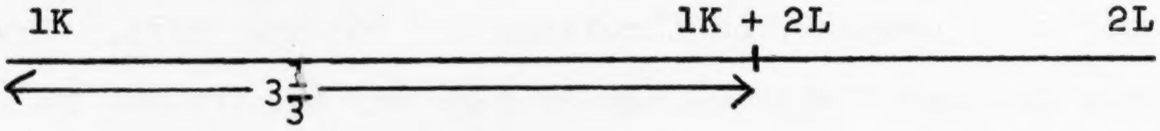
g.


3G
3G + 2H
2H
$1\frac{1}{5}$
$1\frac{4}{5}$

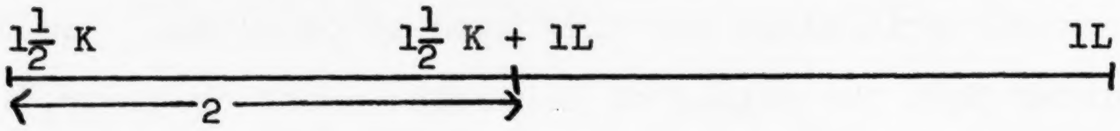
h.


2K
2K + 4L
4L
$3\frac{1}{3}$

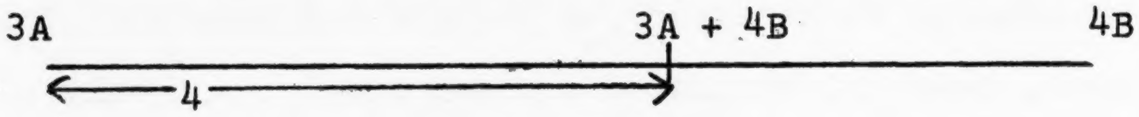
i.


1K
1K + 2L
2L
$3\frac{1}{3}$

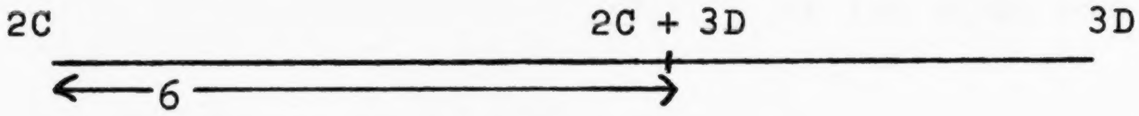
j.


$1\frac{1}{2}$ K
$1\frac{1}{2}$ K + 1L
1L
2

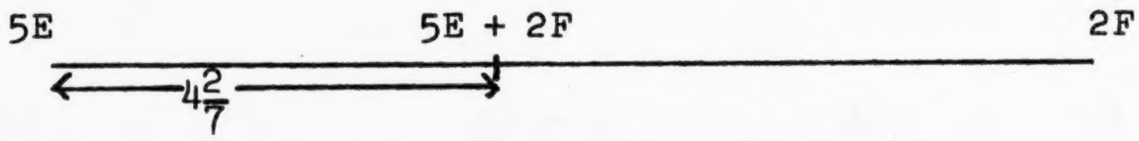
k.



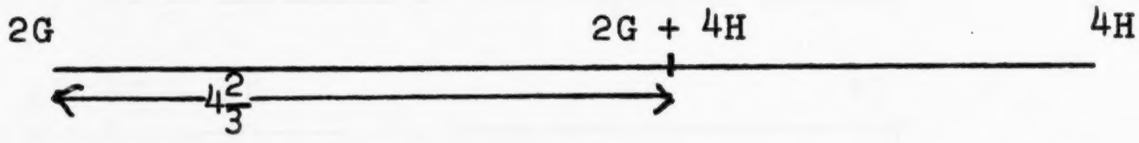
l.



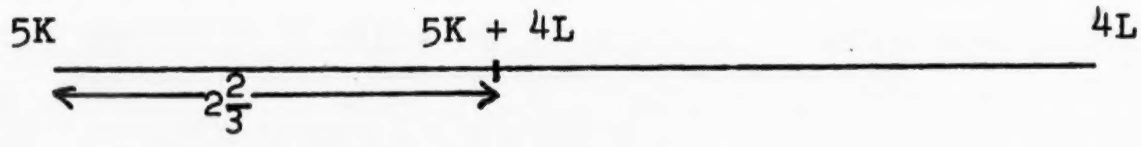
m.


5E
5E + 2F
2F
$4\frac{2}{7}$

n.


2G
2G + 4H
4H
$4\frac{2}{3}$

o.

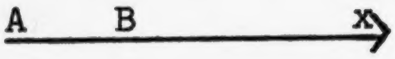

5K
5K + 4L
4L
$2\frac{2}{3}$

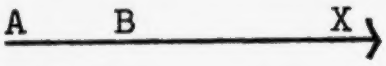
2. a. The center of mass is nearer to B.

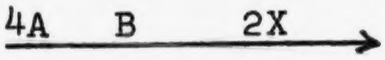
   b. The center of mass is nearer to A.

   c. The center of mass is nearer the point with the greater mass.

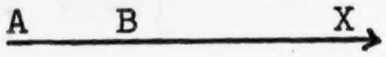
3. a. 2:3 b. 6:1 c. 1:2 d. 1:1

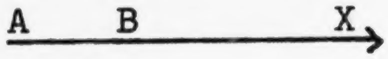
4. a. $3 + x = 4 \Longrightarrow x=1$ AB: BX = 1:3 

   b. $4 + x = 6 \Longrightarrow x=2$ AB: BX = 1:2 

   c. $x + 4 = 6 \Longrightarrow x=2$ AB: BX = 1:2 

   d. $1 + x = 3 \Longrightarrow x=2$ AB: BX = 2:1

   e. $2 + x = 3 \Longrightarrow x=1$ AB: Bx = 1:2 

   f. $x + 9 =12 \Longrightarrow x=3$ AB: Bx = 1:3 

5. a. $b = 12, c = 24$ b. $b = 6, c = 18$

   c. $b = 24, c = 36$ d. $b = 9, c = 21$

6. a. 3 b. 6

7. Let the weights attached to A and B be a and b

   (a) AC: CB = 2:3 = b: a and $a + b = 5$

   then $3b = 2a$, $a = 3$, $b = 2$

   (b) AC: CB = 2:3 = b: a and $a + b = 7$

   then $3b = 2a$, $a = \frac{21}{5}$, $b = \frac{14}{5}$

   (c) AC: CB = 3:4 = b: a and $a + b = 10$

   then $4b = 3a$, solving $a = 5\frac{5}{7}$, $b = 4\frac{2}{7}$

   (d) AC: CB = x: y = b: a and $a + b = 5$

   then $ax = by$

   Solving $a = 5 - b$ so $5x - bx = by$

$bx + by = 5x$ and $b = \frac{5x}{x+y}$

and $a = \frac{5y}{x+y}$

(e) $AC : CB = x : y = b : a$ and $a + b = z$

then $ax = by$

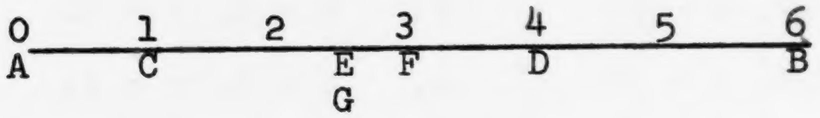
Solving: $a = z - b$ so $xz - xb = by$

and $xz = xb + by$ and $b = \frac{xz}{x+y}$

and $a = \frac{yz}{x+y}$

8. (a-d) 3D + 3C 2B + 3C 1A + 2B

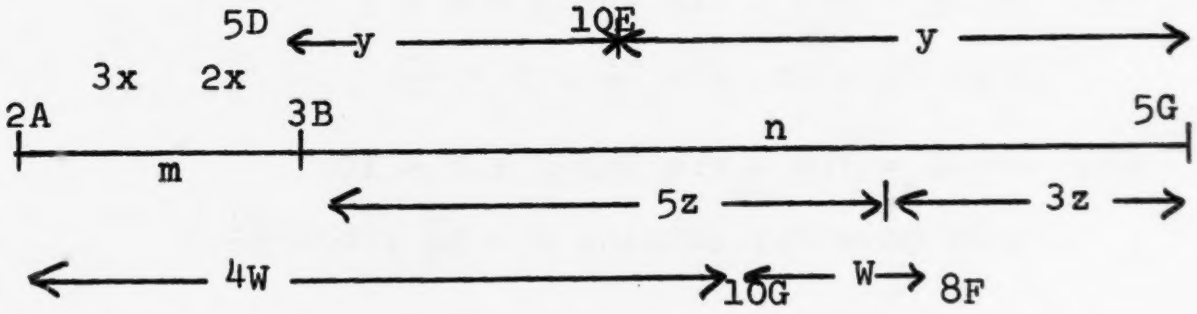
1A + 5F


0
1
2
3
4
5
6
A
C
E
G
F
D
B

(e) Yes

(f) Each triplet resulted in the center E (or G) and weight 6.

9.


5D
y
10E
y
3x
2x
2A
3B
n
5G
m
5z
3z
4W
10G
W
8F

See diagram above.

Let the distance between A and B be m

Let the distance between B and C be n

(a) Adding 2A and 3B we obtain point D with weight

5 and the distances AD and DB can be represented as $3x$ and $2x$ and we have that $5x = m$.

(b) Adding 5D and 5C we obtain point E with weight 10. The distance between D and E and E and C must be equal and we can call them y each.

(c) Adding 3B and 5C we obtain point F with weight 8 and the distances BF and FC can be represented as $5Z$ and $3Z$, and we have $8Z = n$.

(d) Adding 2A and 8F we obtain point G with weight 10 and the distances AG and GF can be represented by $4W$ and W.

(e) We have to show that the distance between A and E is the same as the distance between A and G (or that the distance between E and C is the same as the distance between G and C). This means we have to prove that either $3x + y = 4W$

or $W + 3Z = y$

Now we have from the diagram

$$5x + 8z = m + n \qquad (1)$$

$$5W + 3z = m + n \qquad (2)$$

$$3x + 2y = m + n \qquad (3)$$

From (1) and (3) we get

$$5x + 8z = 3x + 2y$$

or $2x + 8z = 2y$

or $x + 4z = y \qquad (4)$

From (1) and (2) we get

$$5x + 8z = 5W + 3z$$

or $5x + 5z = 5W$

or $x + z = w$ (5)

If we combine (4) and (5)

$$x + 4z = y$$

$$x + z = w$$

we get $3z = y - w$

or $3z + W = y$

which proves that $\overline{EC} = \overline{GC}$ and therefore that points E and G are the same.

(f) This shows that

$$(2A + BC) + 5C = 2A + (3B + 5C).$$

A.5

2. a. Commutation $2B + 1C = 1C + 2B$

b. The Commutation in a and Assonciation.

c. $2B + 3A + 1C = (2B + 3A) + 1C$ (Association)

$= (3A + 2B) + 1C$ Commutation

$= 3A + 2b + 1C$ Association

3. $aA + bB + cC = aA + cC + bB$

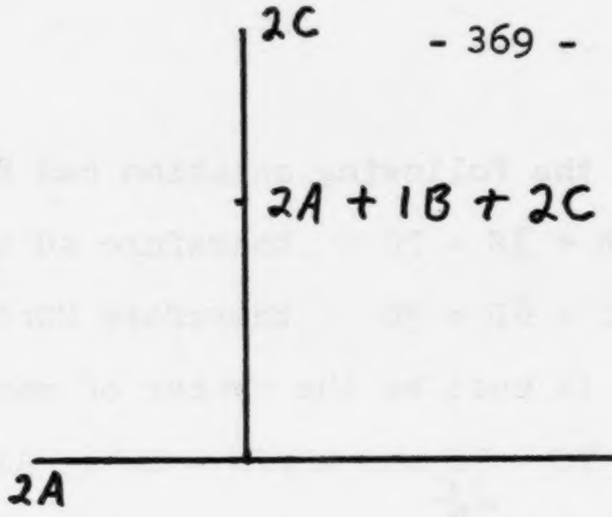
$= bB + aA + cC$

$= bB + cC + aA$

$= cC + aA + bB$

$= cC + bB + aA$

4.

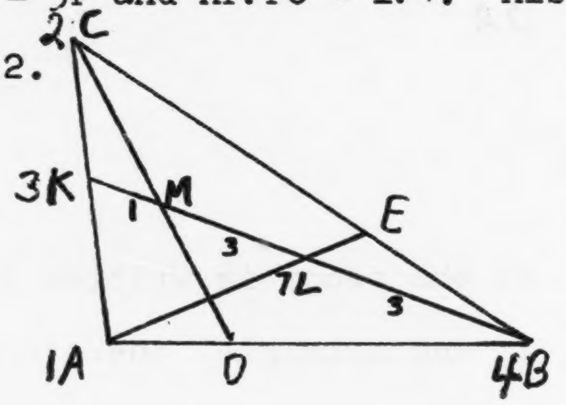


A.7

1. As the proof is written it can apply equally well to any triangle; therefore to all triangles.
2. Failure to carry out the experiment successfully may be due to any of the following or combinations.
    - (a) Line segments are not drawn straight.
    - (b) Line segments are drawn with broad strokes and then the center of mass may not be collinear with related mass points.
    - (c) The ruler was used inaccurately in a measurement.
    - (d) The computation of the ratio of distances to mass points is inaccurate.
3. If medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ meet at G, and AD = 15, BE = 12, and EF = 18, then AG = 10, GD = 5, BG = 8, GE = 4, EG = 12, GF = 6.
4. Using the notation in Exercise 3, if AD = 12, BE = 13, EF = 14, then AG = 8, GD = 4 , BG = $8\frac{2}{3}$, GE = $4\frac{1}{3}$, CG = $9\frac{1}{3}$, GF = $4\frac{2}{3}$.

5. If the hint is accepted the following solution can be given $2B + 1C = 3E$, $4A + 3E = 7G$ therefore AG: GE = 3:4
$4A + 2B = 6D$, $1C + 6D = 7G$ therefore CG: GD = 1:6

6. Since F is in BG and CA it must be the center of masses at C and A. Hence $4A + 1C = 5F$ and AF: FC = 1:4. Also $2B + 5F = 7G$ and BG: GF = 5:2.



7. To solve this problem we use two sets of weights, one to find Kl: LB, the other to find KM: Mb. To find KL: LB use mass points 1A, 2C, 4B. (We are interested in KB ∩ AE = L). Then $1A + 2C = 3K$ and $3K + 4B = 7L$ or KL: LB = 4:3. To find KM: MB, use mass points 2A, 1B, 4C (we are interested in KB ∩ CD = M). Then $2A + 4C = 6K$ and $6K + 1B = 7M$ or KM: MB = 1:6. We may thus think of KB subdivided into 7 segments of the same length. Of these 7, 3 are in BL, 3 are in LM and one is in MK. Therefore Bl = LM = 3MK.

A.9

3. Using the hint we use mass points 2A, LB, 3C, and 1D. $(2A + 1B) + (2C + 1D) = 3E + 3G = 6H$ where H is the midpoint of EG. $(1D + 2A) + (1B + 2C) = 3H + 3F = 6K$ where K is the midpoint of HF. By P_3 and P_4, $6H = 6K$. That is EG and HF bisect each other.

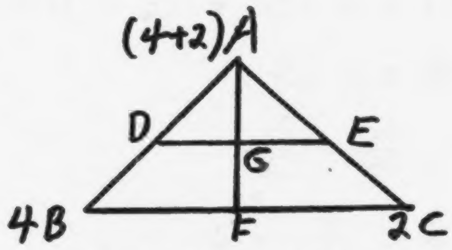
4. Assign weight 3 to P and 3 to R; 1 to S and 1 to Q. Then

$(3P + 1S) + (3R + 1Q) = 4A + 4C = 8E$ where E is the midpoint of AC. Also $(1S + 3R) + (1Q + 3P) = 4B + 4D = 8F$ where F is the midpoint of BD. But $8E = 8F$. Therefore AC and BD bisect each other.

5. Assign weight 2 to C and 2 to A; 1 to B and 2 to D. Then $(2A + 1B) + (2C + 2D) = 3P + 4R = 7E$. Also $(1B + 2C) + (2D + 2A) = 3Q + 4S = 7G$. But $2A + 1B + 2C + 2D = 7F = 7G$. Therefore $F = G$. Thus F is on both Pr and QS and it follows that $F = E$. $4R + 3P = 7E$ RE: EP = 3:4
$4S + 3Q = 7E$ SE: EG = 3:4.

A.11 Answers to Exercises

1. (a)



$$\begin{aligned} &(4B + 4A) + (2A + 2C) \qquad && \text{or} \qquad 4B + 2C + 6A \\ &= 8D + 4E && = 6F + 6A \\ &= 12H. && = 12K. \end{aligned}$$

Therefore $H = K = G$.

Hence, FG: GA = 1:1, and DG: GE = 1:2.

(b - e) Exercises should be deleted as they are confusing and irrelevant.

2. If 4 is assigned to B and 5 to C, then 4 + 5 should be assigned to A in order that D be the center of mass of masses at B and A, and that E be the center of mass of masses at C and A. Thus $4B + 4A = 8D$ and $5C + 5A = 10E$.

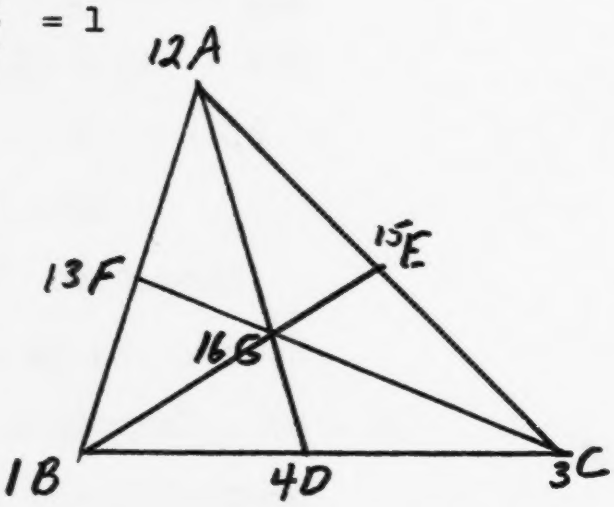
The center of all masses is then $8D + 10E = 18G$ and $DG : GE = 5 : 4$. Also $4B + 5C = 9F + 9A = 18G$. Therefore G is the midpoint of AF.

3. The segment that joins two sides of a triangle at the midpoints bisects any segment joining any point in the third side to the opposite vertex.

4. The segment that joins the trisection point nearer A, intersects any segment joining any point in the third side to A, in the trisection point nearer A.

5. a. Assign weights 1 to B, 3 to C and 12 to A.
   Then, $1B + 3C = 4D$ and $3C + 12A = 15E$.
   Hence, $4D + 12A = 16G$ and $15E + 1B = 16G$.
   and $16G = xF + 3c \Longrightarrow x = 13$.
   Since $1B + 12A = 13F$, $\frac{AB}{FB} = \frac{1}{12}$.

   b. $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{1}{12} \cdot \frac{3}{1} \cdot \frac{4}{1} = 1$

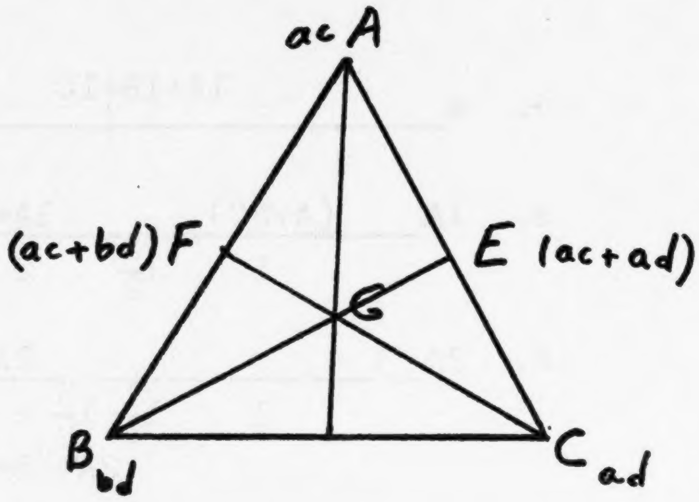


6. Assign 2 to B, 3 to C and 5 to A. Then, $2B + 3C + 5A = 10A$, and $10A = 3C + xF \Longrightarrow x = 7$. Thus,

   $\frac{AF}{FB} = \frac{2}{5}$

   $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{2}{5} \cdot \frac{3}{2} \cdot \frac{5}{3} = 1.$

7. If we assign weights to B, and to C then, to A we should assign $\frac{ac}{d}$, in order that D and E be centers of masses of related mass points. We can simplify computation by


ac A
(ac+bd) F
E (ac+ad)
G
B bd
C ad

multiplying each weight by d. Thus, we assign bd to B, ad to C, and ac to A. This makes F the center of masses at A and B and acA + bdB = (ac + bd)F and $\frac{AF}{FB} = \frac{bd}{ac}$. Then, $\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{bd}{ac} \cdot \frac{a}{b} \cdot \frac{c}{d} = 1$.

8. Using results in Exercise 7 and the equations bdB + adC = (bc + ad)D. Thus, GD: GA = ac : bd + ad.

   Or GD: AD = ac : bd + ad + ac

   That is $\frac{GD}{AD} = \frac{ac}{bd + ad + ac}$.

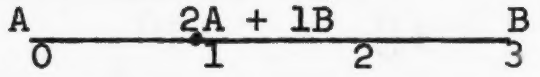
   By similar arguments $\frac{GE}{BE} = \frac{bd}{bd + ad + ac}$

   and $\frac{GF}{CF} = \frac{ad}{bd + ad + ac}$

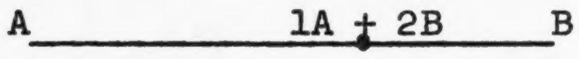
   Hence, $\frac{GD}{AD} + \frac{GE}{BE} + \frac{GF}{CF} = \frac{ac + bd + ad}{bd + ad + ac} = 1$.

A.14 Answers - Review Exercises

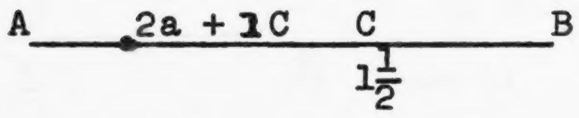
1. a.


A
2A + 1B
B
0
1
2
3

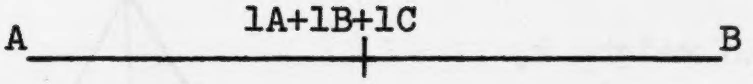
b.



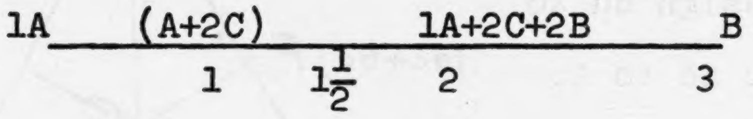
c.


A
2a + 1C
C
B
$1\frac{1}{2}$

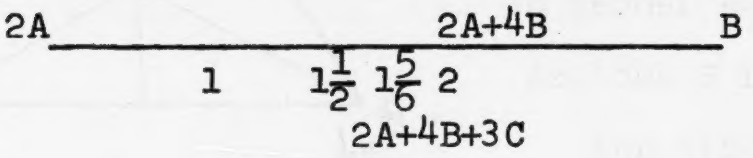
d.



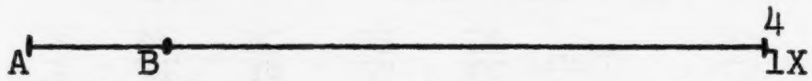
e.


1A
(A+2C)
1A+2C+2B
B
1
1½
2
3

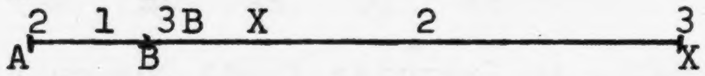
f.


2A
2A+4B
B
1
1½
1 5/6
2
2A+4B+3C

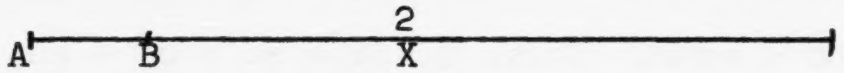
2.a. $x = 1$



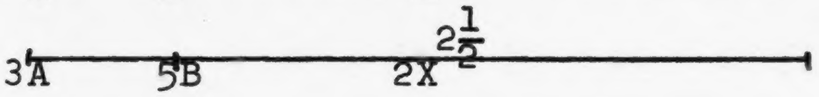
b. $x = 1$


2
1
3B
X
2
3
A
B
X

c. $x = 2$



d. $x = 2$



3.a. 8

b. 4

c. 16

d. $5\frac{1}{3}$

4.a. 1 to C and 4 to A

b. 1B + 1C = 2D,

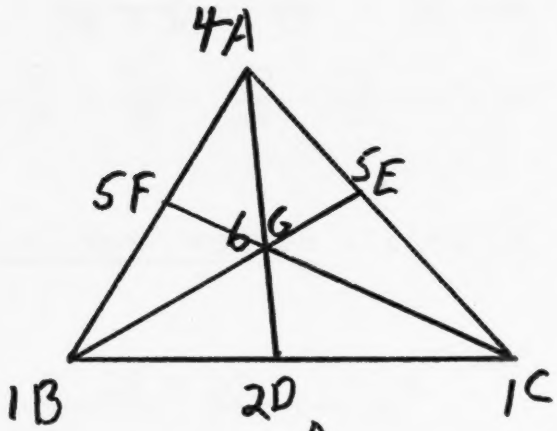
4A + 2D = 6G.

Therefore AG: GD = 1:2

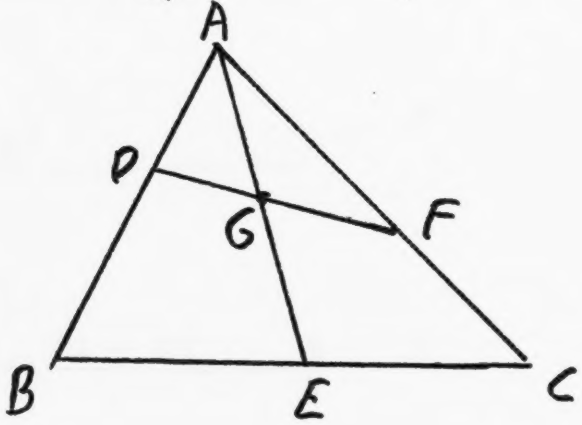
BG: GE = 5:1

c. 4A + 1B = 5F.

Therefore AF: FB = 1:4.


4A
5F
5E
6G
1B
2D
1C

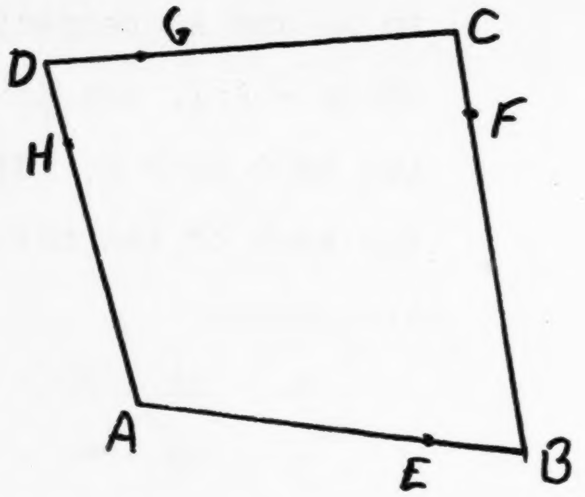
5. If we assign 1 to B, 2 to C then, G is the center of masses A, B and C. If we



assign first 2 to A and then 1 to B by the first assignment D is the center of masses at A and B, and by the second assignment to A, F is the center of masses at A and C.

3A + 1B + 2C = (2A + 1B) + (1A + 2C) = 3D + 3F. Also 3A + 1B + 2C = 3A + (1B + 2C) = 3A + 3E. The center of masses at A, B, C is DF ∩ AE = G. Thus, DG: DF = 1:1 and AG: GE = 1:1. Therefore DF and AE bisect each other.

6. If we assign weights 1 to A, 2 to B, 4 to C and 8 to D then, E, F, G, H are centers of masses at the endpoints of the sides in which each lies. Thus, 1A + 2B = 3E, 2B + 4C = 6F, 4C + 8D = 12G, and 8D + 1A = 9H.



Now, 1A + 2B + 4C + 8D = (1A + 2B) + (4C + 8D) = 3E + 12G or 1A + 2B + 4C + 8D = (2B + 4C) + (8D + 1A) = 6F + 9H. In either case the center of mass of the four masses at A=B=C=D is 15K, where K = HF ∩ GE. Thus, 3E + 12G = 15K and EK: KG = 4:1. Also, 6F + 9H = 15K and FK: KH = 3:2.

Suggested Items for Chapter Test- Chapter 13

1. Draw $\overline{AB}$ making it 4 inches long. On this segment locate the center of the masses for each sum which follows.

a. $3A + 1B$ Call the center C.

b. $1A + 3B$ Call the center D.

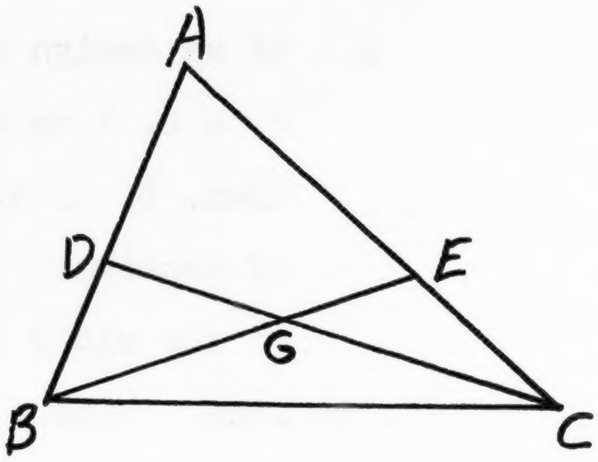
c. $1A + 3B + 4C$ Call it F.

2. For each of the following equations solve for x and compute AB: BX.

   a. $3A + xX = 4B$

   b. $xX + 4B = 7B$

3. In ∧ ABC, points D and E in AB and AC respectively, BD: DA = 2: 1, and AE: EC = 2: 1. Let BE ∩ CD = G. Give a reason for each of the following statements.



   a. $1B + 2A = 3D$.

   b. $2A + 4C = 6E$.

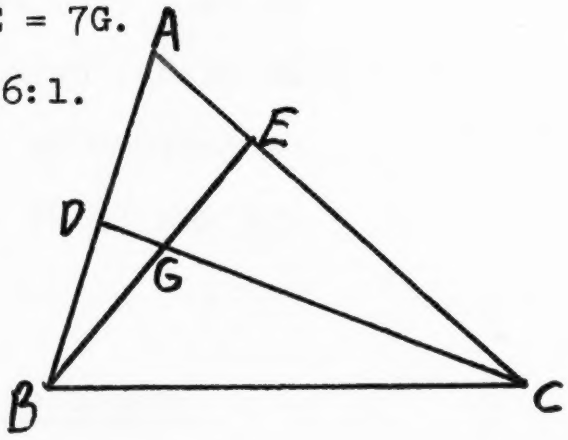
   c. $(1B + 2A) + 4C = 1B + (2A + 4C)$.

   d. $3D + 4C = 1B + 6E$.

   e. $3D + 4C = 7G$ and $1B + 6E = 7G$.

   f. DG: GC = 4: 3 and BG: GE = 6: 1.

4. a. In ∧ ABC, D is in AB and E is in AC. BD: DA = 2: 1. CE: EA = 2: 1. Let BE ∩ CD = G. Prove: BG: GE = 3: 1 and CG: GD = 3: 1



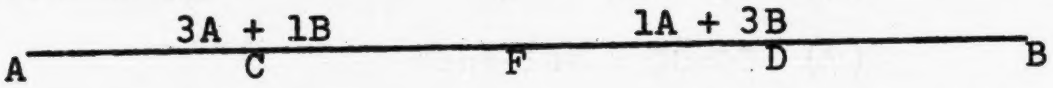
   b. Suppose in ∧ ABC, BD: DA = 3: 1 and CE: EA = 3: 1. Compute BG: GE and CG: GD.

5. In quadrilateral ABCD, E, F, G, H are in AB, BC, CD, DA respectively. E and H are trisection points nearer A and F and G are trisection points nearer C. Prove EFGH is a parallelogram.

Answers for Chapter Test Items

Chapter 13

1.



2. a. $x = 1$, $AB : BX = 1:3$

b. $x = 3$, $AB : BX = 3:4$

3. a. Definition of addition of mass points is satisfied for $1 + 2 = 3$ and $BD : DA = 2:1$.

b. Definition of addition of mass points is satisfied for $2 + 4 = 6$ and $AE : EC = 2:1$.

c. Association (P_3).

d. Substitution Principle, using (a) and (b) in (c).

e. The center of masses at A, B, C must be both on BE and CD, hence - it must be G. (Also $3 + 4 = 1 + 6 = 7$).

f. Definition of addition of mass points is satisfied.

4. a. We assign weights 1 to B, 1 to C and (2 + 2) to A.

(1) $1B + 2A = 3D$ Definition of addition of mass points.

(2) $1C + 2A = 3E$ Same as reason (1).

(3) $(1B + 2A) + 1C = 1B + (2A + 1C)$ P_3.

(4) $3D + 1C = 1B + 3E$ Subsitution Principle (1), (2) in (3).

(5) $3D + 1C = 4G = 1B + 3E$ The center of masses at A, B, C is on BE and CD, that is, at G, and definition of addition.

(6) $1C + 3D = 4G$ Commutation (P_2).

(7) $CG : GD = 3:1$ and $BG : GE = 3:1$ Definition of addition of mass points.

b. $BG : GE = CG : GD = 4:1$

(If $BF : DA = CE : DA = n:1$, then $BG : GE = CG : ED = n+1:1$).

5. Assign weights 2 to A, 2 to C, 1 to B, 1 to D.

(1) $(2A + 1B) + (2C + 1D) = (1B + 2C) + (1D + 2A)$: $P_2 + P_3$.

(2) $2A + 1B = 3E$, $2C + 1D = 3G$, $1B + 2C = 3F$, $1D + 2A = 3H$: Definition of addition of mass points.

(3) $3E + 3G = 3F + 3H$: Subsitution Principle.

(4) $3E + 3G = 6K = 3F + 3H$: Statement (1).

(5) $EK : KG = 1:1$, $FK : KH = 1:1$ Definition of addition of mass points.

(6) EFGH is a parallelogram: If the diagonals of a quadrilateral bisect each other, it is a parallelogram.